From Parametricity to Conservation Laws *via* Noether's Theorem

Robert Atkey

@bentnib

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Polymorphically Typed Lagrangians

$\Downarrow$

Invariance Properties

$\Downarrow$

Conservation Laws for Physical Systems

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Polymorphically Typed Lagrangians

$\Downarrow$ *via Parametricity*

Invariance Properties

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Conservation Laws for Physical Systems

Polymorphically Typed Lagrangians

$\Downarrow$ *via Parametricity*

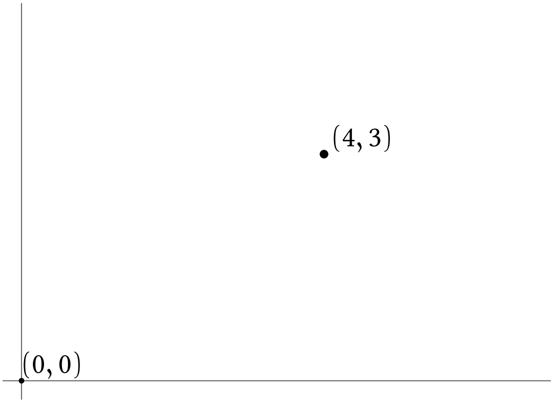
Invariance Properties

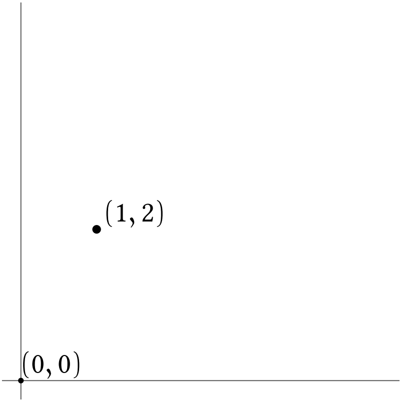
$\Downarrow$ *via Noether's Theorem*

Conservation Laws for Physical Systems

Points and Vectors

(Atkey, Johann, Kennedy POPL2013)


(4, 3)
(0, 0)



(1, 2)
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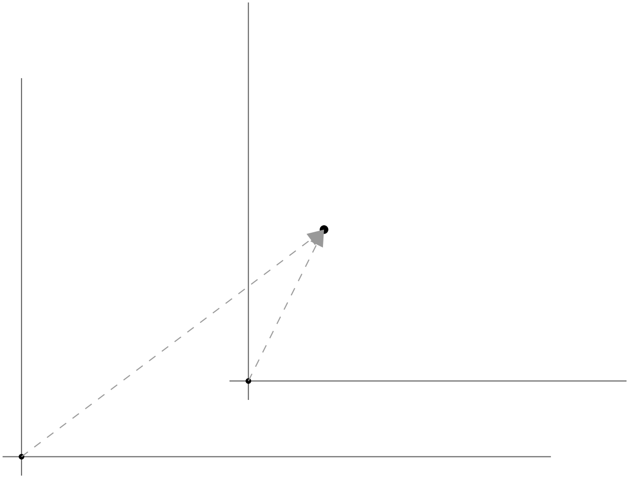

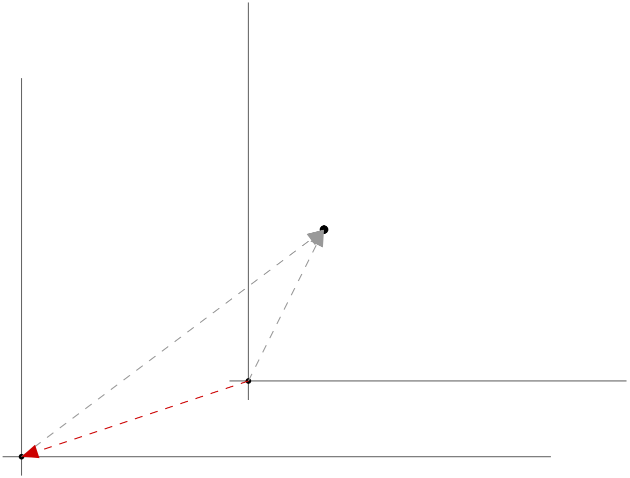

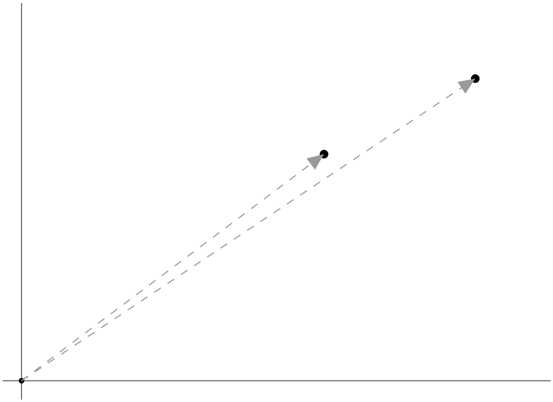

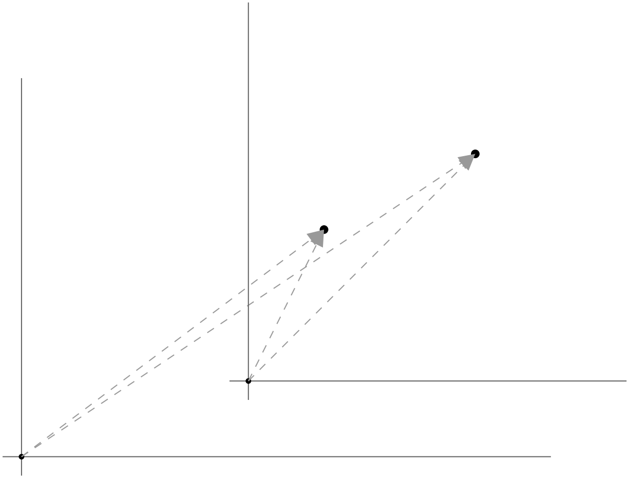

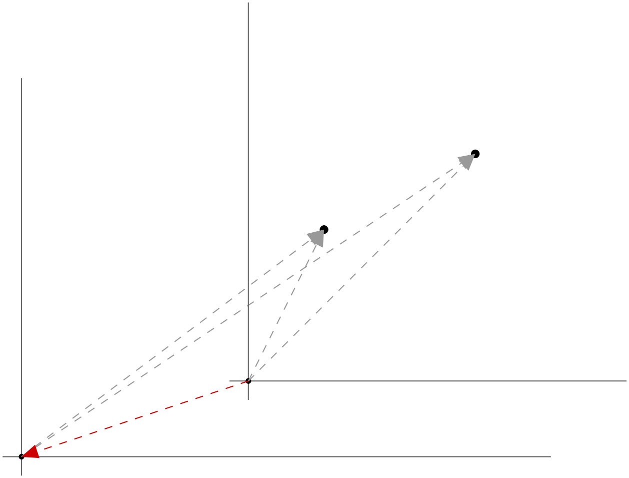

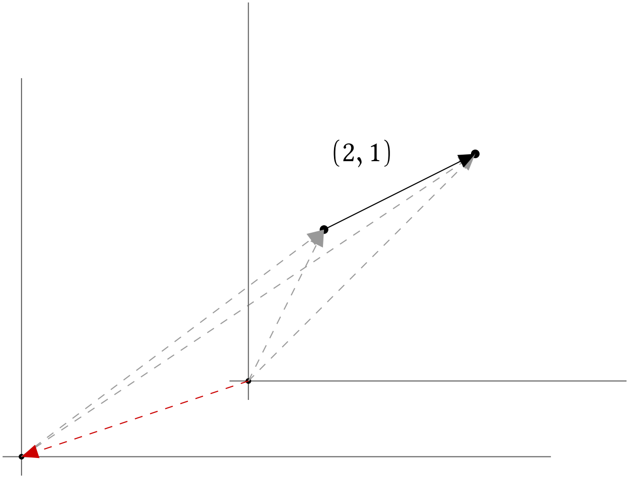
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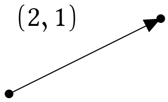
(2, 1)

What is the difference between:

accepting a pair of *points*:

$$f: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

accepting a pair of *vectors*:

$$g: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

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accepting a pair of *vectors*:

$$g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

Points vary with change of origin:

$$\forall t \in T(2).\ \forall \vec{x_1}, \vec{x_2}.\ f(\vec{x_1} + t, \vec{x_2} + t) = f(\vec{x_1}, \vec{x_2})$$

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Idea: *ensure this property using types*

A higher kinded type:

$$\mathbb{R}^2 : \mathsf{T}(2) \longrightarrow *$$

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Different types:

$$f : \forall t{:}\mathsf{T}(2).\, \mathbb{R}^2\langle t\rangle \times \mathbb{R}^2\langle t\rangle \to \mathbb{R}$$

and

$$\mathrm{g} : \mathbb{R}^2\langle 0\rangle \times \mathbb{R}^2\langle 0\rangle \to \mathbb{R}$$

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... with interpretation:

$$\begin{aligned} [\![\mathbb{R}^2]\!]^o\ \cdot &= \mathbb{R} \times \mathbb{R} \\ [\![\mathbb{R}^2]\!]^r\ t &= \{(\vec{v}, \vec{v'}) \mid \vec{v} = \vec{v'} + \vec{t}\} \end{aligned}$$

What can we use invariance properties for?

Lagrangian Mechanics and Noether's Theorem

Lagrangian Mechanics and Noether's Theorem

From Invariance to Conservation Laws

Lagrangian Mechanics and Noether's Theorem

From Invariance to Conservation Laws

Lagrangians:

$$L(t, q, \dot{q}) = T - V$$

where:

T is the total *kinetic energy* of the system

V is the total *potential energy* of the system

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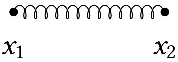
Example:



$$L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2} m(\dot{x}_1{}^2 + \dot{x}_2{}^2) - \frac{1}{2} k(x_1 - x_2)^2$$

Paths:

$$q(t)$$

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The Action:

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q(t), \dot{q}(t))\, dt$$

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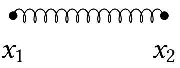
Principle of Stationary Action: *(Euler-Lagrange Equations)*

$$\delta\mathcal{S} = 0 \quad \Leftrightarrow \quad \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{q}} - \tfrac{\partial L}{\partial q} = 0$$

The “physically realisable” paths q satisfy these ODEs

The spring:



$$L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m({\dot{x}_1}^2 + {\dot{x}_2}^2) - \frac{1}{2}k(x_1 - x_2)^2$$

The spring's Equations of Motion:

$$m\ddot{x}_1 = -k(x_1 - x_2) \qquad m\ddot{x}_2 = -k(x_2 - x_1)$$

Newton's second law is *derived*.

Lagrangian Mechanics and Noether's Theorem

From Invariance to Conservation Laws

Given an Action:

$$\mathcal{S}[q;a;b] = \int_a^b L(t, q, \dot{q})\,dt$$

assume transformations of time $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$
assume transformations of space $\Psi_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$
where Φ_0 and Ψ_0 are the identity

Invariance of the Action:

The action is invariant if (for all q, a, b, ϵ):

$$\int_a^b L(t, q(t), \dot{q}(t))\,dt = \int_{\Phi_\epsilon(a)}^{\Phi_\epsilon(b)} L(s, q^*(s), \dot{q}^*(s))\,ds$$

where $q^* = \Psi_\epsilon \circ q \circ \Phi_\epsilon^{-1}$

Noether's (first) Theorem:

If the action

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q, \dot{q})\, dt$$

is invariant under Φ_ϵ and Ψ_ϵ, then

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\psi_i + \left(L - \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)\phi\right) = 0$$

where $\phi = \frac{\partial \Phi}{\partial \epsilon}\Big|_{\epsilon=0}$ and $\psi = \frac{\partial \Psi}{\partial \epsilon}\Big|_{\epsilon=0}$

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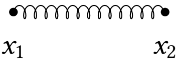
Simplified Invariance:

when $\Phi(t) = t + t'$, and $\Psi(x) = Gx + x'$, then invariance reduces:

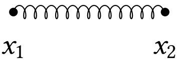
$$L(t, q, \dot{q}) = L(t + t', Gq + x', G\dot{q})$$

The spring:



$$L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m({\dot{x}_1}^2 + {\dot{x}_2}^2) - \frac{1}{2}k(x_1 - x_2)^2$$

The spring:



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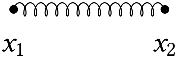
Invariant under change of origin:

$$\forall y.\ L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = L(t, x_1 + y, x_2 + y, \dot{x}_1, \dot{x}_2)$$

The spring:



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$$\forall y.\ L(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = L(t, x_1 + y, x_2 + y, \dot{x}_1, \dot{x}_2)$$

so, for all paths for x_1 and x_2 satisfying the e.o.m.:

$$\frac{d}{dt}m(\dot{x}_1 + \dot{x}_2) = 0$$

Kinds, Types, and Terms for Classical Mechanics

Higher kinded types for Classical Mechanics:

$$\begin{aligned} \mathbb{R}^n &: \mathsf{GL}(n) \longrightarrow \mathsf{T}(n) \longrightarrow \mathsf{CartSp} \\ C^\infty &: \mathsf{CartSp} \longrightarrow \mathsf{CartSp} \longrightarrow * \\ \wr - \int &: \mathsf{CartSp} \longrightarrow * \end{aligned}$$

where

$\mathsf{GL}(n)$	—	the kind of invertible linear transformations
$\mathsf{O}(n)$	—	the kind of orthogonal linear transformations
$\mathsf{T}(n)$	—	the kind of translations
CartSp	—	the kind of cartesian spaces

Term constants for Classical Mechanics:

$$\underline{\vec{c}} \quad : \wr\, \mathbb{R}^n\langle 1, 0\rangle \int$$

$$0 \quad : \forall g{:}\mathsf{GL}(n).\ \wr\, \mathbb{R}^n\langle g, 0\rangle \int$$

$$(+) \quad : \forall g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 + t_2\rangle)$$

$$(-) \quad : \forall g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 - t_2\rangle)$$

$$(*) \quad : \forall g_1{:}\mathsf{GL}(1), g_2{:}\mathsf{GL}(n).\ C^\infty(\mathbb{R}\langle g_1, 0\rangle \times \mathbb{R}^n\langle g_2, 0\rangle, \mathbb{R}\langle \mathrm{scale}_n(g_1) g_2, 0\rangle)$$

$$(\cdot) \quad : \forall g{:}\mathsf{GL}(1), o{:}\mathsf{O}(n).\ C^\infty(\mathbb{R}^n\langle (\mathrm{scale}_n\ g)(\mathrm{ortho}_n\ o), 0\rangle \times \mathbb{R}^n\langle (\mathrm{scale}_n\ g)(\mathrm{ortho}_n\ o), 0\rangle, \mathbb{R}\langle (\mathrm{scale}_n\ g)^2, 0\rangle)$$

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$$
\begin{aligned}
\text{sin} \quad &: \forall z{:}\mathsf{Z}.\ C^{\infty}(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\text{cos} \quad &: \forall z{:}\mathsf{Z}.\ C^{\infty}(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\text{exp} \quad &: \forall t{:}\mathsf{T}(1).\ C^{\infty}(\mathbb{R}\langle 1, t\rangle, \mathbb{R}\langle \exp\ t, 0\rangle) \\
\text{const} &: \wr Y \int \to C^{\infty}(X, Y) \\
\text{id} \quad &: C^{\infty}(X, X) \\
(\ggg) &: C^{\infty}(X, Y) \to C^{\infty}(Y, Z) \to C^{\infty}(X, Z)
\end{aligned}
$$

Group-indexed types yield invariance properties as free theorems

Group-indexed types yield invariance properties as free theorems

(proved using a parametric model built using reflexive graphs)

A syntax for smooth invariant functions:

$$\Theta | \Gamma ; \Delta \vdash M : X$$

where

$\Theta = \alpha_1 : \kappa_1, ..., \alpha_n : \kappa_n$ - kinding context
$\Gamma = z_1 : A_1, ..., z_m : A_m$ - typing context
$\Delta = x_1 : X_1, ..., x_k : X_k$ - spatial context

- Semantics is given by translation into Fω
- $\Theta | \Gamma ; \Delta \vdash M : X \quad \Rightarrow \quad \Theta \vdash \Gamma \vdash \lfloor M \rfloor : C^\infty(\Delta, X)$

Some Classical Mechanical Systems

Free Particle

$$\Theta = t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3)$$
$$\Gamma = \; m : \wr \mathbb{R}\langle 1, 0 \rangle \int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t \rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), t_x \rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0 \rangle$$

$$L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) : \mathbb{R}\langle 1, 0 \rangle$$

Free Particle

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$$L = \frac{1}{2}m(\dot{x}\cdot\dot{x}) : \mathbb{R}\langle 1, 0\rangle$$

Free theorems

$\forall t_t \in \mathbb{R}.\ [\![L]\!](t + t_t, \vec{x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	energy
$\forall \vec{t_x} \in \mathbb{R}^3.\ [\![L]\!](t, \vec{x} + \vec{t_x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	linear momentum
$\forall O \in \mathrm{O}(3).\ [\![L]\!](t, O\vec{x}, O\dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	angular momentum

In detail:

$$\forall O \in \mathrm{O}(3).\ [\![L]\!](t, O\vec{x}, O\dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$$

In particular (rotation around the z-axis):

$$O_\epsilon = \begin{pmatrix} \cos\epsilon & \sin\epsilon & 0 \\ -\sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Apply Noether's theorem with:

$$\Psi_\epsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = O_\epsilon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \cos\epsilon + x_2 \sin\epsilon \\ -x_1 \sin\epsilon + x_2 \cos\epsilon \\ x_3 \end{pmatrix}$$

Derive the conservation law:

$$\frac{d}{dt}(m\dot{x}x_2 - m\dot{x}x_1) = 0$$

Particle in an arbitrary potential field:

$$
\begin{aligned}
\Theta &= t_t : \mathsf{T}(1), o : \mathsf{O}(3) \\
\Gamma &= \; m : \wr \mathbb{R}\langle 1, 0\rangle \int, \\
&\quad\; V : \forall o{:}\mathsf{O}(3).\ C^\infty(\mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle
\end{aligned}
$$

$$L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) - V(x) : \mathbb{R}\langle 1, 0\rangle$$

Conserved Quantities:

- Energy
- Angular momentum

  Even though V is unknown

n-body problem

$$
\begin{aligned}
\Theta &= n : \mathsf{Nat}, t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3) \\
\Gamma &= \ m : \wr \mathbb{R}\langle 1, 0\rangle \int \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, \\
&\quad\; x : \text{vec } n \ (\mathbb{R}^3\langle \text{ortho}_3(o), t_x\rangle), \\
&\quad\; \dot{x} : \text{vec } n \ (\mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle)
\end{aligned}
$$

$$
\begin{aligned}
L = \tfrac{1}{2} m(&\text{sum } (\text{map } (\dot{x}_i.\ \dot{x}_i \cdot \dot{x}_i))\ \dot{x}) - \\
&\text{sum } (\text{map } ((x_i, x_j).\ Gm^2/|x_i - x_j|)\ (\text{cross } x\ x)) : \mathbb{R}\langle 1, 0\rangle
\end{aligned}
$$

n-body problem

$$\begin{aligned}\Theta &= n : \mathsf{Nat}, t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3)\\ \Gamma &= m : \wr \mathbb{R}\langle 1, 0\rangle \int\\ \Delta &= t : \mathbb{R}\langle 1, t_t\rangle,\\ &\quad x : \text{vec } n\ (\mathbb{R}^3\langle \text{ortho}_3(o), t_x\rangle),\\ &\quad \dot{x} : \text{vec } n\ (\mathbb{R}^3\langle \text{ortho}_3(o), 0\rangle)\end{aligned}$$

$$\begin{aligned}L = \tfrac{1}{2} m(&\text{sum } (\text{map } (\dot{x}_i.\ \dot{x}_i \cdot \dot{x}_i))\ \dot{x}) -\\ &\text{sum } (\text{map } ((x_i, x_j).\ Gm^2/|x_i - x_j|)\ (\text{cross } x\ x)) : \mathbb{R}\langle 1, 0\rangle\end{aligned}$$

Conserved quantities:

- Energy
- Linear momentum
- Angular momentum

Pendulum:

$$\begin{aligned}
\Theta &= t_t : \mathsf{T}(1), z : \mathsf{Z} \\
\Gamma &= \ m : \wr\mathbb{R}\langle 1, 0\rangle\int, l : \wr\mathbb{R}\langle 1, 0\rangle\int \\
\Delta &= t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle
\end{aligned}$$

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$$\begin{aligned}
L = \ & \text{let } y = l \sin\theta \text{ in} \\
& \text{let } \dot{x} = l\dot{\theta}\cos\theta \text{ in} \\
& \text{let } \dot{y} = -l\dot{\theta}\sin\theta \text{ in} \\
& \quad \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy : \mathbb{R}\langle 1, 0\rangle
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\end{aligned}$$

Free theorems:

Energy conservation

Invariance under $z : \mathsf{Z}$ not smooth $\Rightarrow$ no conserved property

Damped spring

$$\Theta = t_t : \mathsf{T}(1)$$
$$\Gamma = k : \lbag \mathbb{R}\langle 1, 0\rangle \rbag$$
$$\Delta = t : \mathbb{R}\langle 1, t_t + t_t\rangle, x : \mathbb{R}\langle \exp(-t_t), 0\rangle, \dot{x} : \mathbb{R}\langle \exp(-t_t), 0\rangle$$

$$L = \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\right)\exp(t) : \mathbb{R}\langle 1, 0\rangle$$

Damped spring

$$\begin{aligned}
\Theta &= t_t : \mathsf{T}(1) \\
\Gamma &= \; k : \wr \mathbb{R}\langle 1, 0\rangle \int \\
\Delta &= \; t : \mathbb{R}\langle 1, t_t + t_t\rangle, x : \mathbb{R}\langle \exp(-t_t), 0\rangle, \dot{x} : \mathbb{R}\langle \exp(-t_t), 0\rangle
\end{aligned}$$

$$L = \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\right)\exp(t) : \mathbb{R}\langle 1, 0\rangle$$

Conservation Law:

$$\frac{d}{dt}\left[\left(\frac{1}{2}x\dot{x} + \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right)e^t\right] = 0$$

Summary

In the paper:

- Kinds, types, and terms for Classical Mechanics
- Model using reflexive graph categories
- More examples

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Future work (sober):

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Future work (delusional):

- Quantum Field Theory
- Checking preconditions for using numerical techniques
- Type theoretic reconstruction of the Standard Model?