A Relationally Parametric Model *of* Dependent Type Theory

Robert Atkey
Neil Ghani *(University of Strathclyde)*
Patricia Johann *(Appalachian State University)*

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Relational Parametricity

(Reynolds, 1983)

Type Abstraction

Type Abstraction

$$e : \forall\alpha.\ \alpha \to (\alpha \to \alpha) \to \alpha$$

The implementation e only "knows" two things about α:

- at least one $z : \alpha$ exists;
- and, given one, there is another, by $s : \alpha \to \alpha$.

The program e is uniform under changes of representation of α.

Reynolds' Idea

Formalise e's symmetry via preservation of relations

Relational Parametricity

For example,

$$e : \forall\alpha.\ \alpha \to (\alpha \to \alpha) \to \alpha$$

let X and Y be sets, and let $R \subseteq X \times Y$

if we have $z_1 \in X$, $z_2 \in Y$ such that:

$$(z_1, z_2) \in R$$

and $s_1 : X \to X$, $s_2 : Y \to Y$ such that:

$$\forall(a, b) \in R.\ (s_1\ a, s_2\ b) \in R$$

then

$$(e\ [X]\ z_1\ s_1, e\ [Y]\ z_2\ s_2) \in R$$

Preservation of Relations

implies $(\forall\alpha.\ \alpha \to (\alpha \to \alpha) \to \alpha) \cong \mathbb{N}$

Relational Parametricity

Relational interpretations of types

$$\mathcal{R}[\![\Theta \vdash A]\!]\ \theta\ \theta'\ \rho \subseteq \mathcal{T}[\![\Theta \vdash A]\!]\ \theta \times \mathcal{T}[\![\Theta \vdash A]\!]\ \theta'$$

$$\begin{aligned}
\mathcal{R}[\![\alpha]\!]\ \rho &= \rho(\alpha) \\
\mathcal{R}[\![A \to B]\!]\ \rho &= \{(f_1, f_2) \mid \forall (a_1, a_2) \in \mathcal{R}[\![A]\!]\rho.\ (f_1\ a_1, f_2\ a_2) \in \mathcal{R}[\![B]\!]\rho\} \\
\mathcal{R}[\![\forall\alpha.\ A]\!]\ \rho &= \{(x_1, x_2) \mid\ \ \forall X, Y, R \subseteq X \times Y. \\
&\qquad\qquad (x_1\ [X], x_2\ [Y]) \in \mathcal{R}[\![A]\!](\rho[\alpha \mapsto R])\}
\end{aligned}$$

Relational Parametricity

Identity Extension:

$$\forall x, y \in \mathcal{T}[\![\Theta \vdash A]\!]\theta \quad \Rightarrow \quad ((x, y) \in \mathcal{R}[\![\Theta \vdash A]\!](\mathrm{Eq}_\theta) \Leftrightarrow x = y)$$

and *Abstraction*:

$$\Theta \mid - \vdash e : A \quad \Rightarrow \quad [\![e]\!] \in \mathcal{T}[\![\Theta \vdash A]\!]\theta$$

Routes to Understanding

Denotational Models

Reynolds, Bainbridge-Freyd-Scedrov-Scott, Robinson-Rosolini, Hasegawa, Wadler, Dunphy-Reddy, ...

Operational Models

Pitts, Johann, Ahmed, Birkedal-Møgelberg-Petersen, Dreyer, Vytiniotis-Weirich,...

Logics

Plotkin-Abadi, Birkedal-Møgelberg-Petersen, ...

By Translation

Wadler, Bernardy, ...

Relationally Parametric Models

for

System F

Mutually define base and relational interpretations of types

(Reynolds, 1983) (Bainbridge et al., 1990)

$$
\begin{aligned}
\mathcal{T}[\![\alpha]\!]\theta &= \theta(\alpha) \\
\mathcal{T}[\![A \to B]\!]\theta &= \mathcal{T}[\![A]\!]\theta \to \mathcal{T}[\![B]\!]\theta \\
\mathcal{T}[\![\forall\alpha.A]\!]\theta &= \{\ x : \forall X.\ \mathcal{T}[\![A]\!](\theta[\alpha \mapsto X]) \\
&\qquad \mid\ \forall X, Y, R \subseteq X \times Y. \\
&\qquad\quad \mathcal{R}[\![\tau]\!](\mathrm{Eq}_\theta, \alpha \mapsto R)\ (x\, X)\ (x\, Y)\ \} \\
\mathcal{R}[\![\alpha]\!]\rho &= \rho(\alpha) \\
\mathcal{R}[\![A \to B]\!]\rho &= \{(f_1, f_2) \mid \forall (a_1, a_2) \in \mathcal{R}[\![A]\!]\rho.\ (f_1\ a_1, f_2\ a_2) \in \mathcal{R}[\![B]\!]\rho\} \\
\mathcal{R}[\![\forall\alpha.\tau]\!]\rho\ x\ y &= \{(x_1, x_2) \mid \forall X, Y, R \subseteq X \times Y. \\
&\qquad (x\, X, y\, Y) \in \mathcal{R}[\![\tau]\!](\rho, \alpha \mapsto R)\}
\end{aligned}
$$

$$
\text{then} : \begin{cases} \text{prove Identity Extension} \\ \text{prove Abstraction} \end{cases}
$$

Relational Parametricity *for* Higher Kinds

$(*, * \to *, (* \to *) \to *, \ldots)$

How to interpret kinds?

Implicitly:

$$[\![*]\!] = \text{set} \quad \text{and} \quad [\![*]\!]^R = (X, Y) \mapsto \text{Rel}(X, Y)$$

So let us try:

$$\begin{aligned} [\![*]\!] &= \text{set} \\ [\![\kappa_1 \to \kappa_2]\!] &= [\![\kappa_1]\!] \to [\![\kappa_2]\!] \end{aligned}$$

and

$$\begin{aligned} [\![\kappa]\!]^R &: [\![\kappa]\!] \times [\![\kappa]\!] \to \text{set} \\ [\![*]\!]^R &= (X, Y) \mapsto \text{Rel}(X, Y) \\ [\![\kappa_1 \to \kappa_2]\!]^R &= (F, G) \mapsto \forall X, Y. [\![\kappa_1]\!]^R(X, Y) \to [\![\kappa_2]\!]^R(FX, GY) \end{aligned}$$

Identity extension?

Recall identity extension:

$$\forall x, y \in \mathcal{T}[\![\Theta \vdash A : *]\!]\theta \quad \Rightarrow \quad ((x, y) \in \mathcal{R}[\![\Theta \vdash A : *]\!](\mathrm{Eq}_{\theta}) \Leftrightarrow x = y)$$

What is “equality” for $F : * \to *$?

No good answer in general.

Solution:

Build-in an “identity” for every semantic type operator

Every semantic type operator’s identity preserves identities

Kinds as Reflexive Graphs

Reflexive Graph Categories

(Hasegawa, 1994)

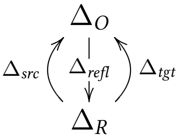
(Robinson and Rosolini, 1994)

(Dunphy and Reddy, 2004)

Let $RG = \bullet \underset{\delta_1}{\overset{\delta_0}{\leftleftarrows}} \overset{i}{\rightarrow} \bullet$ such that $\delta_0 \circ i = id$ and $\delta_1 \circ i = id$.

Interpret kinds as elements of Set^{RG}.

Kinds as "Categories without Composition"



Kinds as Reflexive Graphs

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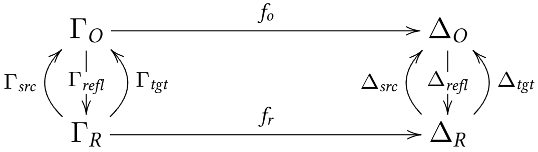

Kinds as Reflexive Graphs

Reflexive Graph Categories

(Hasegawa, 1994)
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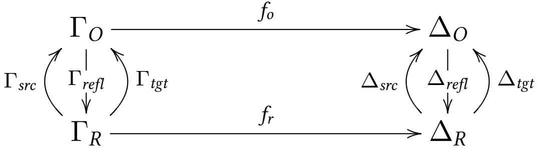
Let $RG = \bullet \begin{matrix}\xleftarrow{\delta_0} \\ \xrightarrow{i} \\ \xleftarrow{\delta_1}\end{matrix} \bullet$ such that $\delta_0 \circ i = id$ and $\delta_1 \circ i = id$.

Interpret kinds as elements of Set^{RG}.

Kinds as "Categories without Composition"



Higher kinds are interpreted using the cartesian-closed structure.

Interpretation of System Fω

Interpretation of Base Kind

$$
\begin{aligned}
[\![*]\!]_O &= \text{set} \\
[\![*]\!]_R &= \{(X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set}\} \\
[\![*]\!]_{refl}(X) &= (X, X, \text{Eq}_X) \\
[\![*]\!]_{src}(X, Y, R) &= X \\
[\![*]\!]_{tgt}(X, Y, R) &= Y
\end{aligned}
$$

Interpretation of Types $\Theta \vdash A : \kappa$

— interpreted as a morphism in Set^{RG}

— recreates the mutual induction used for System F

Interpretation of Terms $\Theta \mid \Gamma \vdash e : A$

— interpreted as a natural transformations “without composition”

— yields the standard abstraction theorem

Interpretation of System Fω

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\end{aligned}
$$

Interpretation of Types and Terms

the categories $\text{Set}^{\text{RG}}(\Delta, [\![*]\!])$

— objects are “semantic types”

— morphisms are “semantic terms”

Dependent Types

Dependent Types

Types depend on terms

$$\Pi A : \mathsf{U}.\ \Pi n : \mathsf{nat}.\ \mathsf{T}\ (\mathrm{Vec}\ A\ n) \to \mathsf{T}\ (\mathrm{Vec}\ A\ n)$$

Types computed from Terms

$$\begin{aligned} \mathrm{Vec} &: \mathsf{U} \to \mathsf{nat} \to \mathsf{U} \\ \mathrm{Vec} &= \lambda A\ n.\ \mathsf{natrec}(x.\ \mathsf{U}, \mathsf{Unit}, x\ p.\ A \times p, n) \end{aligned}$$

Martin-Löf Type Theory

(Martin-Löf, 1984)

- Π-types, natural numbers
- Tarski-style universe (U, T) of small types
  - closed under Π and natural numbers
  - (optionally impredicative)

Relationally Parametric Models
of
Dependent Types

Models *of* Dependent Types

Families Fibration

$$\begin{array}{c} \mathrm{Fam}(\mathrm{Set}) \\ \Big\downarrow{\scriptstyle p} \\ \mathrm{Set} \end{array}$$

Families

Objects of Fam(Set): $(X \in \mathrm{Set}, A \in X \to \mathrm{Set})$

- Types: $\Gamma \vdash A$ type
  - $X \in \mathrm{Set}$ models the context Γ;
  - $A \in X \to \mathrm{Set}$ models the type A.
- Terms: $\Gamma \vdash e : A$
  - Morphisms $(X, \lambda x.\ 1) \to (X, A)$ in Fam(Set)

(*Families Fibration*)$^{\mathrm{RG}}$

$$\begin{array}{c} \mathrm{Fam}(\mathrm{Set})^{\mathrm{RG}} \\ \Big\downarrow{\scriptstyle p^{\mathrm{RG}}} \\ \mathrm{Set}^{\mathrm{RG}} \end{array}$$

Families of Reflexive Graphs

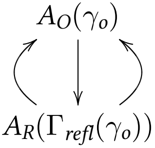
For a reflexive graph Γ, a *family of reflexive graphs A over* Γ:

$$\begin{array}{ll}
A_O & \in \Gamma_O \to \mathrm{Set} \\
A_R & \in \Gamma_R \to \mathrm{Set} \\
A_{refl} & \in \forall \gamma_o \in \Gamma_O.\ A_O(\gamma_o) \to A_R(\Gamma_{refl}(\gamma_o)) \\
A_{src} & \in \forall \gamma_r \in \Gamma_R.\ A_R(\gamma_r) \to A_O(\Gamma_{src}(\gamma_r)) \\
A_{tgt} & \in \forall \gamma_r \in \Gamma_R.\ A_R(\gamma_r) \to A_O(\Gamma_{tgt}(\gamma_r))
\end{array}$$

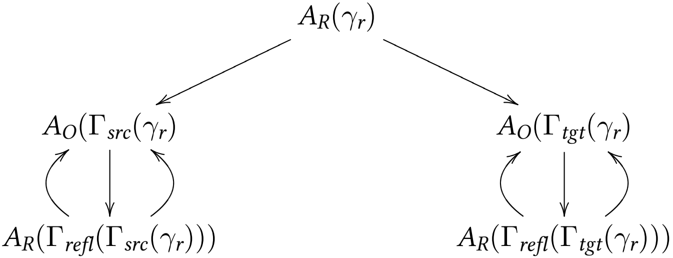
RG-Fam(Γ): the category of reflexive graph familes over Γ

Families of Reflexive Graphs

For every $\gamma_o \in \Gamma_O$, a reflexive graph:



For every $\gamma_r \in \Gamma_R$, a relation between reflexive graphs:



From System F(ω) types to Families

The Interpretation of Base Kind:

$$
\begin{aligned}
[\![*]\!]_O &= \mathrm{set} \\
[\![*]\!]_R &= \{(X, Y, R \subseteq X \times Y) \mid X, Y \in \mathrm{set}\}
\end{aligned}
$$

From System F(ω) types to Families

The Interpretation of Base Kind:

$$\begin{aligned} [\![*]\!]_O &= \text{set} \\ [\![*]\!]_R &= \{(X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set}\} \end{aligned}$$

For Semantic Types: $A \in \text{Set}^{\text{RG}}(\Gamma, [\![*]\!])$,

for all $\gamma_o \in \Gamma_O$, $A_O(\gamma_o)$ is a small set

for all $\gamma_r \in \Gamma_R$, $A_R(\gamma_r)$ is a triple:

$$(A_O(\Gamma_{src}(\gamma_r)), A_O(\Gamma_{tgt}(\gamma_r)), R \subseteq A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)))$$

From System F(ω) types to Families

The Interpretation of Base Kind:

$$\begin{aligned} [\![*]\!]_O &= \text{set} \\ [\![*]\!]_R &= \{(X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set}\} \end{aligned}$$

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$$(A_O(\Gamma_{src}(\gamma_r)), A_O(\Gamma_{tgt}(\gamma_r)), R \subseteq A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)))$$

In terms of Families of Reflexive Graphs: $A \in \text{RG-Fam}(\Gamma)$ is:

- *small*, if $A_O(\gamma_o)$ and $A_R(\gamma_r)$ are small sets;
- *discrete*, if $(A_O(\gamma_o), A_R(\Gamma_{refl}(\gamma_o)))$ is iso. to (X, X) for some X;
- *proof-irrelevant*, if
  $$A_R(\gamma_r) \to A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)) \text{ is injective}$$

Representing System F(ω) types

Small, discrete, proof-irrelevant families

$\text{RG-Fam}_{\text{stpi}}(\Gamma)$

Representing System F(ω) types

Small, discrete, proof-irrelevant families

$\text{RG-Fam}_{\text{stpi}}(\Gamma)$

Representation

$\text{Set}^{\text{RG}}(\Gamma, [\![*]\!]) \simeq \text{RG-Fam}_{\text{stpi}}(\Gamma)$

Universes

Rules

$$\frac{}{\Gamma \vdash \mathsf{U}\ \text{type}} \qquad \frac{\Gamma \vdash M : \mathsf{U}}{\Gamma \vdash \mathsf{T}(M)\ \text{type}}$$

$$\frac{}{\Gamma \vdash \mathsf{nat} : \mathsf{U}} \qquad \frac{\Gamma \vdash M : \mathsf{U} \qquad \Gamma, x : \mathsf{T}(M) \vdash N : \mathsf{U}}{\Gamma \vdash \Pi x : M.\ N : \mathsf{U}}$$

Interpretation of the universe U

$$\begin{aligned} U_O(\gamma_o) &= \text{small discrete reflexive graphs} \\ U_R(\gamma_r) &= \{(X, Y, R, R_{src}, R_{tgt}) \mid \langle R_{src}, R_{tgt}\rangle : R \to X_O \times Y_O \text{ is injective}\} \end{aligned}$$

$T \in \text{RG-Fam}(\Gamma.U)$:

$$\begin{aligned} T_O(\gamma_o, (X_O, X_R)) &= X_O \\ T_R(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) &= R \\ T_{refl}(\gamma_o, (X_O, X_R)) &= X_{refl} \\ T_{src}(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) &= R_{src} \\ T_{tgt}(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) &= R_{tgt} \end{aligned}$$

Natural Numbers

As a family of reflexive graphs:

$$\begin{aligned} \mathrm{nat}_O(\gamma_o) &= \mathbb{N} \\ \mathrm{nat}_R(\gamma_r) &= \mathbb{N} \end{aligned}$$

Structure:

- Easy to define zero, succ, natrec
- The family nat is small, discrete and proof-irrelevant

Π-types

Π-types

Objects

$$
\begin{aligned}
&(\Pi AB)_O(\gamma_o) = \\
&\quad \{\ (f_o, f_r) \mid \\
&\qquad f_o \in \forall a_o \in A_O(\gamma_o).\ B_O(\gamma_o, a_o), \\
&\qquad f_r \in \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)).\ B_R(\Gamma_{refl}(\gamma_o), a_r), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\quad B_{src}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{src}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\quad B_{tgt}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{tgt}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_o \in A_O(\gamma_o).\ B_{refl}(\gamma_o, a_o)(f_o\ a_o) = f_r(A_{refl}(\gamma_o)(a_o))\ \}
\end{aligned}
$$

- Transformer on objects

Π-types

Objects

$$
\begin{aligned}
&(\Pi A B)_O(\gamma_o) = \\
&\quad \{ (f_o, f_r) \mid \\
&\qquad f_o \in \forall a_o \in A_O(\gamma_o).\ B_O(\gamma_o, a_o), \\
&\qquad f_r \in \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)).\ B_R(\Gamma_{refl}(\gamma_o), a_r), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\quad B_{src}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{src}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\quad B_{tgt}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{tgt}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_o \in A_O(\gamma_o).\ B_{refl}(\gamma_o, a_o)(f_o\ a_o) = f_r(A_{refl}(\gamma_o)(a_o)) \ \}
\end{aligned}
$$

- Transformer on objects
- Transformer on relations

Π-types

Objects

$$
\begin{aligned}
&(\Pi A B)_O(\gamma_o) = \\
&\quad \{\ (f_o, f_r)\ | \\
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&\qquad f_r \in \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)).\ B_R(\Gamma_{refl}(\gamma_o), a_r), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\qquad B_{src}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{src}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
&\qquad\qquad B_{tgt}(\Gamma_{refl}(\gamma_o), a_r)(f_r\ a_r) = f_o(A_{tgt}(\Gamma_{refl}(\gamma_o))(a_r)), \\
&\qquad \forall a_o \in A_O(\gamma_o).\ B_{refl}(\gamma_o, a_o)(f_o\ a_o) = f_r(A_{refl}(\gamma_o)(a_o))\ \}
\end{aligned}
$$

- Transformer on objects
- Transformer on relations
- Source and targets agree

Π-types

Objects

$$
\begin{aligned}
&(\Pi A B)_O(\gamma_o) = \\
&\quad \{\ (f_o, f_r) \mid \\
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&\qquad\quad \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
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\end{aligned}
$$

- Transformer on objects
- Transformer on relations
- Source and targets agree
- Reflexive relations are preserved

Π-types

Relations

$$
\begin{aligned}
&(\Pi A B)_R(\gamma_r) = \\
&\quad \{\ ((f_o^{src}, f_r^{src}), (f_o^{tgt}, f_r^{tgt}), r) \mid \\
&\qquad (f_o^{src}, f_r^{src}) \in (\Pi A B)_O(\Gamma_{src}(\gamma_r)), \\
&\qquad (f_o^{tgt}, f_r^{tgt}) \in (\Pi A B)_O(\Gamma_{tgt}(\gamma_r)), \\
&\qquad r \in \forall a_r \in A_R(\gamma_r).\ B_R(\gamma_r, a_r), \\
&\qquad \forall a_r \in A_R(\gamma_r).\ B_{src}(\gamma_r, a_r)(r\ a_r) = f_o^{src}(A_{src}(\gamma_r)(a_r)), \\
&\qquad \forall a_r \in A_R(\gamma_r).\ B_{tgt}(\gamma_r, a_r)(r\ a_r) = f_o^{tgt}(A_{tgt}(\gamma_r)(a_r))\ \}
\end{aligned}
$$

Π-types

Relations

$$
\begin{aligned}
&(\Pi A B)_R(\gamma_r) = \\
&\quad \{ ((f_o^{src}, f_r^{src}), (f_o^{tgt}, f_r^{tgt}), r) \mid \\
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&\qquad \forall a_r \in A_R(\gamma_r).\ B_{tgt}(\gamma_r, a_r)(r\ a_r) = f_o^{tgt}(A_{tgt}(\gamma_r)(a_r)) \}
\end{aligned}
$$

- Source and target Π-objects

Π-types

Relations

$$
\begin{aligned}
&(\Pi A B)_R(\gamma_r) = \\
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&\qquad (f_o^{tgt}, f_r^{tgt}) \in (\Pi A B)_O(\Gamma_{tgt}(\gamma_r)), \\
&\qquad r \in \forall a_r \in A_R(\gamma_r).\ B_R(\gamma_r, a_r), \\
&\qquad \forall a_r \in A_R(\gamma_r).\ B_{src}(\gamma_r, a_r)(r\ a_r) = f_o^{src}(A_{src}(\gamma_r)(a_r)), \\
&\qquad \forall a_r \in A_R(\gamma_r).\ B_{tgt}(\gamma_r, a_r)(r\ a_r) = f_o^{tgt}(A_{tgt}(\gamma_r)(a_r)) \}
\end{aligned}
$$

- Source and target Π-objects
- Relation transformer

Π-types

Relations

$$
\begin{aligned}
&(\Pi A B)_R(\gamma_r) = \\
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&\qquad (f_o^{tgt}, f_r^{tgt}) \in (\Pi A B)_O(\Gamma_{tgt}(\gamma_r)), \\
&\qquad r \in \forall a_r \in A_R(\gamma_r).\ B_R(\gamma_r, a_r), \\
&\qquad \forall a_r \in A_R(\gamma_r).\ B_{src}(\gamma_r, a_r)(r\ a_r) = f_o^{src}(A_{src}(\gamma_r)(a_r)), \\
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\end{aligned}
$$

- Source and target Π-objects
- Relation transformer
- Sources and targets agree

Dependent Products

Sound

This interpretation of Π-types is sound

- for β- and η-equality
- for general reasons
- so it is unique up to isomorphism

Small, discrete, proof-irrelevant

If $B \in \text{RG-Fam}(\Gamma.A)$ is discrete and proof-irrelevant,

- then so is ΠAB

If A and B are small, then so is ΠAB

- if "set" is impredicative, then only B need be small

Classical Mechanics' kinds as reflexive graphs:

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$[\![\mathsf{CartSp}]\!] = (\mathbb{N}, \text{diffeomorphisms on } \mathbb{R}^n, \mathrm{id})$
Diffeomorphisms are smooth functions with smooth inverses

Applications *of* Relational Parametricity *for* Dependent Types

A Free Theorem

A Polymorphic Function

$$\Gamma \vdash M : \Pi a : \mathsf{U}.\ \mathsf{T}(a) \to \mathsf{T}(a)$$

Free Theorem given:

- $\Gamma \vdash X : \mathsf{U}$
- $\Gamma \vdash Y : \mathsf{U}$
- $\Gamma \vdash f : T(X) \to T(Y)$
- $\Gamma \vdash x : T(X)$

we have the semantically justified axiom:

$$\Gamma \vdash f\,(M\,X\,x) = M\,Y\,(f\,x) : T(Y)$$

- Crucially use proof-irrelevance

Indexed Initial Algebras

(omitting the universe decoder T)

Specification

For *functors* $(F : (X \to \mathsf{U}) \to (X \to \mathsf{U}), \mathit{fmap}_F)$, $\mu F : X \to \mathsf{U}$, with

$$\begin{aligned}
\mathrm{in}_F &: \Pi x : X.\ F(\mu F)x \to (\mu F)x \\
\mathrm{fold}_F &: \Pi A : X \to \mathsf{U}.(\Pi x : X.\ FAx \to Ax) \to (\Pi x : X.\ (\mu F)x \to Ax)
\end{aligned}$$

with β- and η-laws

Implementation

$$\begin{aligned}
\mu F &= \lambda x.\Pi A : X \to \mathsf{U}.(\Pi z : X.\ FAz \to Az) \to Ax \\
\mathit{fold}_F &= \lambda A.\lambda f.\lambda x.\lambda e.\ e\ A\ f \\
\mathit{in}_F &= \lambda x.\lambda e.\lambda A.\lambda f.\ f\ A\ (\mathit{fmap}_F\ (\mu F)\ A\ (\mathit{fold}_F\ A\ f)\ x\ e)
\end{aligned}$$

use relational parametricity to prove the η-law

Summary

Relationally parametric model of Dependent Types

- Contexts as reflexive graphs
- Types as families of reflexive graphs

Applications of Dependently-Typed Parametricity

- Free Theorems
- Initial Algebras for Indexed Types

Future work

- Relationship with Homotopy Types?
- Higher Dimensions?
- Internalisation?
- Universe Hierarchy?
- Final coalgebras