A Relationally Parametric Model of Dependent Type Theory

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Relational Parametricity

(Reynolds, 1983)
Type Abstraction

\[ e : \forall \alpha. \; \alpha \to (\alpha \to \alpha) \to \alpha \]

The implementation \( e \) only “knows” two things about \( \alpha \):

- at least one \( z : \alpha \) exists;
- and, given one, there is another, by \( s : \alpha \to \alpha \).

The program \( e \) is uniform under changes of representation of \( \alpha \).

Reynolds’ Idea

Formalise \( e \)’s symmetry via preservation of relations
Relational Parametricity

For example,

\[ e : \forall \alpha. \, \alpha \to (\alpha \to \alpha) \to \alpha \]

let \( X \) and \( Y \) be sets, and let \( R \subseteq X \times Y \)

if we have \( z_1 \in X, \, z_2 \in Y \) such that:

\[ (z_1, z_2) \in R \]

and \( s_1 : X \to X, \, s_2 : Y \to Y \) such that:

\[ \forall (a, b) \in R. \, (s_1 a, s_2 b) \in R \]

then

\[ (e [X] z_1 s_1, e [Y] z_2 s_2) \in R \]

Preservation of Relations

implies \( (\forall \alpha. \, \alpha \to (\alpha \to \alpha) \to \alpha) \cong \mathbb{N} \)
Relational Parametricity

Relational interpretations of types

$$
\mathcal{R}[\Theta \vdash A] \theta \theta' \rho \subseteq \mathcal{T}[\Theta \vdash A] \theta \times \mathcal{T}[\Theta \vdash A] \theta'
$$

$$
\mathcal{R}[\alpha] \rho = \rho(\alpha)
\mathcal{R}[A \to B] \rho = \{(f_1, f_2) \mid \forall (a_1, a_2) \in \mathcal{R}[A]\rho. (f_1 a_1, f_2 a_2) \in \mathcal{R}[B]\rho\}
\mathcal{R}[\forall \alpha. A] \rho = \{(x_1, x_2) \mid \forall X, Y, R \subseteq X \times Y.
\quad (x_1 [X], x_2 [Y]) \in \mathcal{R}[A](\rho[\alpha \mapsto R])\}
$$

Relational Parametricity

Identity Extension:

$$
\forall x, y \in \mathcal{T}[\Theta \vdash A]\theta \quad \Rightarrow \quad ((x, y) \in \mathcal{R}[\Theta \vdash A](\text{Eq}_\theta) \iff x = y)
$$

and Abstraction:

$$
\Theta \mid - \vdash e : A \quad \Rightarrow \quad [e] \in \mathcal{T}[\Theta \vdash A]\theta
$$
Routes to Understanding

**Denotational Models**
Reynolds, Bainbridge-Freyd-Scedrov-Scott, Robinson-Rosolini, Hasegawa, Wadler, Dunphy-Reddy, ...

**Operational Models**
Pitts, Johann, Ahmed, Birkedal-Møgelberg-Petersen, Dreyer, Vytiniotis-Weirich,...

**Logics**
Plotkin-Abadi, Birkedal-Møgelberg-Petersen, ...

**By Translation**
Wadler, Bernardy, ...
Relationally Parametric Models for System F
Mutually define base and relational interpretations of types

(Reynolds, 1983) (Bainbridge et al., 1990)

\[ \mathcal{T}[^{\alpha}]\theta = \theta(\alpha) \]
\[ \mathcal{T}[A \rightarrow B]\theta = \mathcal{T}[A]\theta \rightarrow \mathcal{T}[B]\theta \]
\[ \mathcal{T}[\forall \alpha.A]\theta = \{ x : \forall X. \mathcal{T}[A](\theta[\alpha \mapsto X]) \mid \forall X, Y, R \subseteq X \times Y. \mathcal{R}[\tau](\text{Eq}_\theta, \alpha \mapsto R) (x X) (x Y) \} \]
\[ \mathcal{R}[\alpha]\rho = \rho(\alpha) \]
\[ \mathcal{R}[A \rightarrow B]\rho = \{ (f_1, f_2) \mid \forall (a_1, a_2) \in \mathcal{R}[A]\rho. (f_1 a_1, f_2 a_2) \in \mathcal{R}[B]\rho \} \]
\[ \mathcal{R}[\forall \alpha.\tau] \rho x y = \{ (x_1, x_2) \mid \forall X, Y, R \subseteq X \times Y. \]
\[ (x X, y Y) \in \mathcal{R}[\tau](\rho, \alpha \mapsto R) \} \]

then : \{ prove Identity Extension prove Abstraction \}
Relational Parametricity

_for_

Higher Kinds

(*, * → *, (* → *) → *, ...)
How to interpret kinds?

Implicitly:

\[[\ast]\] = \text{set} \quad \text{and} \quad \left[\ast\right]^R = (X, Y) \mapsto \text{Rel}(X, Y)

So let us try:

\left[\ast\right] = \text{set}
\left[\kappa_1 \rightarrow \kappa_2\right] = \left[\kappa_1\right] \rightarrow \left[\kappa_2\right]

and

\left[\kappa\right]^R : \left[\kappa\right] \times \left[\kappa\right] \rightarrow \text{set}
\left[\ast\right]^R = (X, Y) \mapsto \text{Rel}(X, Y)
\left[\kappa_1 \rightarrow \kappa_2\right]^R = (F, G) \mapsto \forall X, Y.\left[\kappa_1\right]^R(X, Y) \rightarrow \left[\kappa_2\right]^R(FX, GY)
Identity extension?

Recall identity extension:

\[ \forall x, y \in \mathcal{T}[\Theta \vdash A : \ast] \theta \implies ((x, y) \in \mathcal{R}[\Theta \vdash A : \ast](\text{Eq}_\theta) \iff x = y) \]

What is “equality” for \( F : \ast \to \ast \)?

No good answer in general.

Solution:

Build-in an “identity” for every semantic type operator
Every semantic type operator’s identity preserves identities
Kinds as Reflexive Graphs

Reflexive Graph Categories
(Hasegawa, 1994)
(Robinson and Rosolini, 1994)
(Dunphy and Reddy, 2004)

Let $RG = \bullet \xleftarrow{\delta_0} \xrightarrow{\delta_1} \bullet$ such that $\delta_0 \circ i = id$ and $\delta_1 \circ i = id$.

Interpret kinds as elements of $\text{Set}^{RG}$.

Kinds as “Categories without Composition”

\[
\begin{array}{c}
\Delta_O \\
\Delta_{src} \xleftarrow{\Delta_{refl}} \xrightarrow{\Delta_{tgt}} \Delta_{tgt} \\
\Delta_R
\end{array}
\]
Kinds as Reflexive Graphs

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Interpret kinds as elements of \( \text{Set}^{RG} \).

Kinds as “Categories without Composition”

\[
\begin{array}{ccc}
\Gamma_O & \xrightarrow{f_o} & \Delta_O \\
\Gamma_{\text{src}} & \xrightarrow{\Gamma_{\text{refl}}} & \Gamma_{\text{tgt}} \\
\Gamma_R & \xrightarrow{f_r} & \Delta_R \\
\Delta_{\text{src}} & \xrightarrow{\Delta_{\text{refl}}} & \Delta_{\text{tgt}}
\end{array}
\]
Kinds as Reflexive Graphs

**Reflexive Graph Categories**
(Hasegawa, 1994)
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Let $RG = \bullet \xleftarrow{\delta_0} \xrightarrow{\delta_1} \bullet$ such that $\delta_0 \circ i = id$ and $\delta_1 \circ i = id$.

Interpret kinds as elements of $\text{Set}^{RG}$.

**Kinds as “Categories without Composition”**

![Diagram](image)

Higher kinds are interpreted using the cartesian-closed structure.
Interpretation of System Fω

**Interpretation of Base Kind**

\[
\begin{align*}
\llbracket * \rrbracket_O & = \text{set} \\
\llbracket * \rrbracket_R & = \{ (X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set} \} \\
\llbracket * \rrbracket_{\text{refl}}(X) & = (X, X, \text{Eq}_X) \\
\llbracket * \rrbracket_{\text{src}}(X, Y, R) & = X \\
\llbracket * \rrbracket_{\text{tgt}}(X, Y, R) & = Y
\end{align*}
\]

**Interpretation of Types**

\[\Theta \vdash A : \kappa\]

— interpreted as a morphism in Set\(^{RG}\)

— recreates the mutual induction used for System F

**Interpretation of Terms**

\[\Theta \mid \Gamma \vdash e : A\]

— interpreted as a natural transformations “without composition”

— yields the standard abstraction theorem
Interpretation of System Fω

Interpretation of Base Kind

\[
\begin{align*}
\llbracket * \rrbracket O &= \text{set} \\
\llbracket * \rrbracket R &= \{ (X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set} \} \\
\llbracket * \rrbracket refl(X) &= (X, X, \text{Eq}_X) \\
\llbracket * \rrbracket src(X, Y, R) &= X \\
\llbracket * \rrbracket tgt(X, Y, R) &= Y
\end{align*}
\]

Interpretation of Types and Terms
the categories \( \text{Set}^{RG}(\Delta, \llbracket * \rrbracket) \)
— objects are “semantic types”
— morphisms are “semantic terms”
Dependent Types
Dependent Types

Types depend on terms

\[ \Pi A : U. \, \Pi n : \text{nat.} \, T (\text{Vec} \, A \, n) \rightarrow T (\text{Vec} \, A \, n) \]

Types computed from Terms

\[ \text{Vec} : U \rightarrow \text{nat} \rightarrow U \]
\[ \text{Vec} = \lambda A \, n. \, \text{natrec}(x. \, U, \, \text{Unit}, \, x \, p. \, A \times p, \, n) \]

Martin-Löf Type Theory
(Martin-Löf, 1984)

– \(\Pi\)-types, natural numbers
– Tarski-style universe \((U, T)\) of small types
  – closed under \(\Pi\) and natural numbers
  – (optionally impredicative)
Relationally Parametric Models of Dependent Types
Models of Dependent Types

Families Fibration

\[ \text{Fam}(\text{Set}) \]

\[ \xrightarrow{p} \]

\[ \text{Set} \]

Families

Objects of Fam(Set): \((X \in \text{Set}, A \in X \rightarrow \text{Set})\)
- Types: \(\Gamma \vdash A \text{ type}\)
  - \(X \in \text{Set}\) models the context \(\Gamma\);
  - \(A \in X \rightarrow \text{Set}\) models the type \(A\).
- Terms: \(\Gamma \vdash e : A\)
  - Morphisms \((X, \lambda x. 1) \rightarrow (X, A)\) in Fam(Set)
Relationally Parametric Models of Dependent Types

\[(Families Fibration)^{\text{RG}}\]

\[
\begin{array}{c}
\text{Fam(Set)}^{\text{RG}} \\
\downarrow^{p^{\text{RG}}} \\
\text{Set}^{\text{RG}}
\end{array}
\]

Families of Reflexive Graphs

For a reflexive graph $\Gamma$, a family of reflexive graphs $A$ over $\Gamma$:

\[
\begin{align*}
A_O & \in \Gamma_O \to \text{Set} \\
A_R & \in \Gamma_R \to \text{Set} \\
A_{\text{refl}} & \in \forall \gamma_o \in \Gamma_O. \; A_O(\gamma_o) \to A_R(\Gamma_{\text{refl}}(\gamma_o)) \\
A_{\text{src}} & \in \forall \gamma_r \in \Gamma_R. \; A_R(\gamma_r) \to A_O(\Gamma_{\text{src}}(\gamma_r)) \\
A_{\text{tgt}} & \in \forall \gamma_r \in \Gamma_R. \; A_R(\gamma_r) \to A_O(\Gamma_{\text{tgt}}(\gamma_r))
\end{align*}
\]

$\text{RG-Fam}(\Gamma)$: the category of reflexive graph families over $\Gamma$
Families of Reflexive Graphs

For every $\gamma_o \in \Gamma_O$, a reflexive graph:

\[
A_O(\gamma_o) \\
\downarrow \\
A_R(\Gamma_{refl}(\gamma_o))
\]

For every $\gamma_r \in \Gamma_R$, a relation between reflexive graphs:

\[
A_R(\gamma_r) \\
A_O(\Gamma_{src}(\gamma_r)) \\
\downarrow \\
A_R(\Gamma_{refl}(\Gamma_{src}(\gamma_r))) \\
A_O(\Gamma_{tgt}(\gamma_r)) \\
\downarrow \\
A_R(\Gamma_{refl}(\Gamma_{tgt}(\gamma_r)))
\]
From System F(ω) types to Families

The Interpretation of Base Kind:

\[
\begin{align*}
[\star]_O & = \text{set} \\
[\star]_R & = \{(X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set}\}
\end{align*}
\]
From System F(\(\omega\)) types to Families

**The Interpretation of Base Kind:**

\[
[\ast]_O = \text{set} \\
[\ast]_R = \{ (X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set} \}
\]

**For Semantic Types:** \(A \in \text{Set}^{RG}(\Gamma, [\ast])\),
for all \(\gamma_o \in \Gamma_O\), \(A_O(\gamma_o)\) is a small set
for all \(\gamma_r \in \Gamma_R\), \(A_R(\gamma_r)\) is a triple:

\[
(A_O(\Gamma_{src}(\gamma_r)), A_O(\Gamma_{tgt}(\gamma_r)), R \subseteq A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)))
\]
From System F(ω) types to Families

**The Interpretation of Base Kind:**

\[
\begin{align*}
\lfloor * \rfloor_O &= \text{set} \\
\lfloor * \rfloor_R &= \{ (X, Y, R \subseteq X \times Y) \mid X, Y \in \text{set} \}
\end{align*}
\]

**For Semantic Types:** \( A \in \text{Set}^{\text{RG}}(\Gamma, \lfloor * \rfloor) \),
for all \( \gamma_o \in \Gamma_O \), \( A_O(\gamma_o) \) is a small set
for all \( \gamma_r \in \Gamma_R \), \( A_R(\gamma_r) \) is a triple:

\[
(A_O(\Gamma_{src}(\gamma_r)), A_O(\Gamma_{tgt}(\gamma_r)), R \subseteq A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)))
\]

**In terms of Families of Reflexive Graphs:** \( A \in \text{RG-Fam}(\Gamma) \) is:

- **small**, if \( A_O(\gamma_o) \) and \( A_R(\gamma_r) \) are small sets;
- **discrete**, if \( (A_O(\gamma_o), A_R(\Gamma_{refl}(\gamma_o))) \) is iso. to \( (X, X) \) for some \( X \);
- **proof-irrelevant**, if
  \[
  A_R(\gamma_r) \to A_O(\Gamma_{src}(\gamma_r)) \times A_O(\Gamma_{tgt}(\gamma_r)) \text{ is injective}
  \]
Representing System $F(\omega)$ types

Small, discrete, proof-irrelevant families

$\text{RG-Fam}_{\text{stpi}}(\Gamma)$
Representing System F(\(\omega\)) types

*Small, discrete, proof-irrelevant families*

\(\text{RG-Fam}_{\text{stpi}}(\Gamma)\)

*Representation*

\(\text{Set}^{\text{RG}}(\Gamma, [\ast]) \simeq \text{RG-Fam}_{\text{stpi}}(\Gamma)\)
Universes

Rules

\[ \Gamma \vdash U : \text{type} \]
\[ \Gamma \vdash M : U \]
\[ \Gamma \vdash T(M) : \text{type} \]
\[ \Gamma \vdash M : U \]
\[ \Gamma, x : T(M) \vdash N : U \]
\[ \Gamma \vdash \Pi x : M. N : U \]

Interpretation of the universe \( U \)

\[ U_O(\gamma_0) = \text{small discrete reflexive graphs} \]
\[ U_R(\gamma_r) = \{ (X, Y, R, R_{src}, R_{tgt}) \mid \langle R_{src}, R_{tgt} \rangle : R \to X_O \times Y_O \text{ is injective} \} \]

\[ T \in \text{RG-Fam}(\Gamma, U) : \]
\[ T_O(\gamma_0, (X_O, X_R)) = X_O \]
\[ T_R(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) = R \]
\[ T_{refl}(\gamma_0, (X_O, X_R)) = X_{refl} \]
\[ T_{src}(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) = R_{src} \]
\[ T_{tgt}(\gamma_r, (X, Y, R, R_{src}, R_{tgt})) = R_{tgt} \]
Natural Numbers

As a family of reflexive graphs:

\[
\begin{align*}
\text{nat}_O(\gamma_o) &= \mathbb{N} \\
\text{nat}_R(\gamma_r) &= \mathbb{N}
\end{align*}
\]

Structure:

- Easy to define zero, succ, natrec
- The family nat is small, discrete and proof-irrelevant
Π-types

- transformer on objects
- transformer on relations
- source and targets agree
- reflexive relations are preserved
\( \Pi\text{-types} \)

**Objects**

\[
(\Pi A B)_O(\gamma_0) = \{ (f_o, f_r) | \\
  f_o \in \forall a_o \in A_O(\gamma_0). B_O(\gamma_o, a_o), \\
  f_r \in \forall a_r \in A_R(\Gamma_{\text{refl}}(\gamma_0)). B_R(\Gamma_{\text{refl}}(\gamma_0), a_r), \\
  \forall a_r \in A_R(\Gamma_{\text{refl}}(\gamma_0)). \\
  B_{\text{src}}(\Gamma_{\text{refl}}(\gamma_0), a_r)(f_r a_r) = f_o(A_{\text{src}}(\Gamma_{\text{refl}}(\gamma_0))(a_r)), \\
  \forall a_r \in A_R(\Gamma_{\text{refl}}(\gamma_0)). \\
  B_{\text{tgt}}(\Gamma_{\text{refl}}(\gamma_0), a_r)(f_r a_r) = f_o(A_{\text{tgt}}(\Gamma_{\text{refl}}(\gamma_0))(a_r)), \\
  \forall a_o \in A_O(\gamma_0). B_{\text{refl}}(\gamma_0, a_o)(f_o a_o) = f_r(A_{\text{refl}}(\gamma_0)(a_o)) \}
\]

- Transformer on objects
\[ (\Pi A B)_O(\gamma_o) = \{ (f_o, f_r) \mid \]
\[ f_o \in \forall a_o \in A_O(\gamma_o). B_O(\gamma_o, a_o), \]
\[ f_r \in \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). B_R(\Gamma_{refl}(\gamma_o), a_r), \]
\[ \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \]
\[ B_{src}(\Gamma_{refl}(\gamma_o), a_r)(f_r a_r) = f_o(A_{src}(\Gamma_{refl}(\gamma_o))(a_r)), \]
\[ \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \]
\[ B_{tgt}(\Gamma_{refl}(\gamma_o), a_r)(f_r a_r) = f_o(A_{tgt}(\Gamma_{refl}(\gamma_o))(a_r)), \]
\[ \forall a_o \in A_O(\gamma_o). B_{refl}(\gamma_o, a_o)(f_o a_o) = f_r(A_{refl}(\gamma_o)(a_o)) \} \]

- Transformer on objects
- Transformer on relations
Π-types

Objects

\((\Pi \! AB)_O(\gamma_o) = \{ (f_o, f_r) | \)
\[
 f_o \in \forall a_o \in A_O(\gamma_o). B_O(\gamma_o, a_o), \\
 f_r \in \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). B_R(\Gamma_{refl}(\gamma_o), a_r), \\
 \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
 B_{src}(\Gamma_{refl}(\gamma_o), a_r)(f_r a_r) = f_o(A_{src}(\Gamma_{refl}(\gamma_o))(a_r)), \\
 \forall a_r \in A_R(\Gamma_{refl}(\gamma_o)). \\
 B_{tgt}(\Gamma_{refl}(\gamma_o), a_r)(f_r a_r) = f_o(A_{tgt}(\Gamma_{refl}(\gamma_o))(a_r)), \\
 \forall a_o \in A_O(\gamma_o). B_{refl}(\gamma_o, a_o)(f_o a_o) = f_r(A_{refl}(\gamma_o)(a_o)) \} \]

▶ Transformer on objects
▶ Transformer on relations
▶ Source and targets agree
Π-types

Objects

\[(\Pi_{o} A B)_{O}(\gamma_{o}) = \{ (f_{o}, f_{r}) \mid \]
\[f_{o} \in \forall a_{o} \in A_{O}(\gamma_{o}). B_{O}(\gamma_{o}, a_{o}), \]
\[f_{r} \in \forall a_{r} \in A_{R}(\Gamma_{refl}(\gamma_{o})). B_{R}(\Gamma_{refl}(\gamma_{o}), a_{r}), \]
\[\forall a_{r} \in A_{R}(\Gamma_{refl}(\gamma_{o})). B_{src}(\Gamma_{refl}(\gamma_{o}), a_{r})(f_{r} a_{r}) = f_{o}(A_{src}(\Gamma_{refl}(\gamma_{o}))(a_{r})), \]
\[\forall a_{r} \in A_{R}(\Gamma_{refl}(\gamma_{o})). B_{tgt}(\Gamma_{refl}(\gamma_{o}), a_{r})(f_{r} a_{r}) = f_{o}(A_{tgt}(\Gamma_{refl}(\gamma_{o}))(a_{r})), \]
\[\forall a_{o} \in A_{O}(\gamma_{o}). B_{refl}(\gamma_{o}, a_{o})(f_{o} a_{o}) = f_{r}(A_{refl}(\gamma_{o})(a_{o})) \} \]

- Transformer on objects
- Transformer on relations
- Source and targets agree
- Reflexive relations are preserved
π-types

Relations

\[(\Pi \text{AB})_R(\gamma_r) = \]
\[\{ ((f^\text{src}_o, f^\text{src}_r), (f^\text{tgt}_o, f^\text{tgt}_r), r) | \]
\[ (f^\text{src}_o, f^\text{src}_r) \in (\Pi \text{AB})_O(\Gamma_{\text{src}}(\gamma_r)), \]
\[ (f^\text{tgt}_o, f^\text{tgt}_r) \in (\Pi \text{AB})_O(\Gamma_{\text{tgt}}(\gamma_r)), \]
\[ r \in \forall a_r \in A_R(\gamma_r). B_R(\gamma_r, a_r), \]
\[ \forall a_r \in A_R(\gamma_r). B_{\text{src}}(\gamma_r, a_r)(r a_r) = f^\text{src}_o(A_{\text{src}}(\gamma_r)(a_r)), \]
\[ \forall a_r \in A_R(\gamma_r). B_{\text{tgt}}(\gamma_r, a_r)(r a_r) = f^\text{tgt}_o(A_{\text{tgt}}(\gamma_r)(a_r)) \} \]
Π-types

Relations

\[
(\Pi A B)_{R}(\gamma_r) = \{ ((f^{src}_o, f^{src}_r), (f^{tgt}_o, f^{tgt}_r), r) \mid (f^{src}_o, f^{src}_r) \in (\Pi A B)_{O}(\Gamma_{src}(\gamma_r)), (f^{tgt}_o, f^{tgt}_r) \in (\Pi A B)_{O}(\Gamma_{tgt}(\gamma_r)), \]
\[
r \in \forall a_r \in A_{R}(\gamma_r). B_{R}(\gamma_r, a_r), \forall a_r \in A_{R}(\gamma_r). B_{src}(\gamma_r, a_r)(r a_r) = f^{src}_o(A_{src}(\gamma_r)(a_r)), \forall a_r \in A_{R}(\gamma_r). B_{tgt}(\gamma_r, a_r)(r a_r) = f^{tgt}_o(A_{tgt}(\gamma_r)(a_r)) \}
\]

- Source and target Π-objects
(\Pi AB)_R(\gamma_r) =
\{ ((f^\text{src}_o, f^\text{src}_r), (f^\text{tgt}_o, f^\text{tgt}_r), r) |
(f^\text{src}_o, f^\text{src}_r) \in (\Pi AB)_O(\Gamma_{\text{src}}(\gamma_r)),
(f^\text{tgt}_o, f^\text{tgt}_r) \in (\Pi AB)_O(\Gamma_{\text{tgt}}(\gamma_r)),
\forall a_r \in A_R(\gamma_r). B_R(\gamma_r, a_r),
\forall a_r \in A_R(\gamma_r). B_{\text{src}}(\gamma_r, a_r)(r a_r) = f^\text{src}_o(A_{\text{src}}(\gamma_r)(a_r)),
\forall a_r \in A_R(\gamma_r). B_{\text{tgt}}(\gamma_r, a_r)(r a_r) = f^\text{tgt}_o(A_{\text{tgt}}(\gamma_r)(a_r)) \}

- Source and target \(\Pi\)-objects
- Relation transformer
$$\Pi$-types

Relations

$$(\Pi AB)_R(\gamma_r) = $$

$$\{ ((f^\text{src}_o, f^\text{src}_r), (f^t\text{gt}_o, f^t\text{gt}_r), r) |$$

$$(f^\text{src}_o, f^\text{src}_r) \in (\Pi AB)_O(\Gamma_{\text{src}}(\gamma_r)),$$
$$(f^t\text{gt}_o, f^t\text{gt}_r) \in (\Pi AB)_O(\Gamma_{\text{tgt}}(\gamma_r)),$$

$$r \in \forall a_r \in A_R(\gamma_r). B_R(\gamma_r, a_r),$$

$$\forall a_r \in A_R(\gamma_r). B_{\text{src}}(\gamma_r, a_r)(r \ a_r) = f^\text{src}_o(A_{\text{src}}(\gamma_r)(a_r)),$$

$$\forall a_r \in A_R(\gamma_r). B_{\text{tgt}}(\gamma_r, a_r)(r \ a_r) = f^t\text{gt}_o(A_{\text{tgt}}(\gamma_r)(a_r)) \}$$

- Source and target $\Pi$-objects
- Relation transformer
- Sources and targets agree
Dependent Products

**Sound**
This interpretation of \( \Pi \)-types is sound
- for \( \beta \)- and \( \eta \)-equality
- for general reasons
- so it is unique up to isomorphism

**Small, discrete, proof-irrelevant**
If \( B \in \text{RG-Fam}(\Gamma.A) \) is discrete and proof-irrelevant,
- then so is \( \Pi A B \)

If \( A \) and \( B \) are small, then so is \( \Pi A B \)
- if “set” is impredicative, then only \( B \) need be small
Classical Mechanics’ kinds as reflexive graphs:

$$[\text{GL}(n)] = ([*], \text{GL}(n), I)$$

GL(n) is the group of invertible linear transformations on $$\mathbb{R}^n$$
Classical Mechanics’ kinds as reflexive graphs:

\[ \mathcal{GL}(n) = (\{\ast\}, \text{GL}(n), I) \]
GL(n) is the group of invertible linear transformations on \( \mathbb{R}^n \)

\[ \mathcal{O}(n) = (\{\ast\}, \text{O}(n), I) \]
O(n) is the group of orthogonal transformations on \( \mathbb{R}^n \)
Classical Mechanics’ kinds as reflexive graphs:

$$\mathcal{GL}(n) = (\{\ast\}, \text{GL}(n), I)$$

GL(n) is the group of invertible linear transformations on \(\mathbb{R}^n\)

$$\mathcal{O}(n) = (\{\ast\}, \text{O}(n), I)$$

O(n) is the group of orthogonal transformations on \(\mathbb{R}^n\)

$$\mathcal{T}(n) = (\{\ast\}, \text{T}(n), 0)$$

T(n) is the group of translations on \(\mathbb{R}^n\)
Classical Mechanics’ kinds as reflexive graphs:

\[ \begin{align*}
\mathbb{GL}(n) & = (\{\ast\}, \mathbb{GL}(n), I) \\
\text{GL}(n) & \text{ is the group of invertible linear transformations on } \mathbb{R}^n
\end{align*} \]

\[ \begin{align*}
\mathbb{O}(n) & = (\{\ast\}, \mathbb{O}(n), I) \\
\text{O}(n) & \text{ is the group of orthogonal transformations on } \mathbb{R}^n
\end{align*} \]

\[ \begin{align*}
\mathbb{T}(n) & = (\{\ast\}, \mathbb{T}(n), 0) \\
\text{T}(n) & \text{ is the group of translations on } \mathbb{R}^n
\end{align*} \]

\[ \begin{align*}
\mathbb{Z} & = (\{\ast\}, \mathbb{Z}, 0) \\
\mathbb{Z} & \text{ is the additive group of integers}
\end{align*} \]
Classical Mechanics’ kinds as reflexive graphs:

\[ \square \text{GL}(n) = (\{\ast\}, \text{GL}(n), I) \]

\( \text{GL}(n) \) is the group of invertible linear transformations on \( \mathbb{R}^n \)

\[ \square \text{O}(n) = (\{\ast\}, \text{O}(n), I) \]

\( \text{O}(n) \) is the group of orthogonal transformations on \( \mathbb{R}^n \)

\[ \square \text{T}(n) = (\{\ast\}, \text{T}(n), 0) \]

\( \text{T}(n) \) is the group of translations on \( \mathbb{R}^n \)

\[ \square \mathbb{Z} = (\{\ast\}, \mathbb{Z}, 0) \]

\( \mathbb{Z} \) is the additive group of integers

\[ \square \text{CartSp} = (\mathbb{N}, \text{diffeomorphisms on } \mathbb{R}^n, \text{id}) \]

Diffeomorphisms are smooth functions with smooth inverses
Applications of Relational Parametricity for Dependent Types
A Free Theorem

A Polymorphic Function

\[ \Gamma \vdash M : \Pi a : U. \ T(a) \rightarrow T(a) \]

Free Theorem given:

- \( \Gamma \vdash X : U \)
- \( \Gamma \vdash Y : U \)
- \( \Gamma \vdash f : T(X) \rightarrow T(Y) \)
- \( \Gamma \vdash x : T(X) \)

we have the semantically justified axiom:

\[ \Gamma \vdash f(M \ X \ x) = M \ Y \ (f \ x) : T(Y) \]

- Crucially use proof-irrelevance
Indexed Initial Algebras

(omitting the universe decoder T)

Specification

For functors \((F : (X \rightarrow U) \rightarrow (X \rightarrow U), \text{fmap}_F)\), \(\mu F : X \rightarrow U\), with

\[ \text{in}_F : \Pi x : X. F(\mu F)x \rightarrow (\mu F)x \]

\[ \text{fold}_F : \Pi A : X \rightarrow U. (\Pi x : X. FAx \rightarrow Ax) \rightarrow (\Pi x : X. (\mu F)x \rightarrow Ax) \]

with \(\beta\) - and \(\eta\)-laws

Implementation

\[ \mu F = \lambda x.\Pi A : X \rightarrow U. (\Pi z : X. FAz \rightarrow Az) \rightarrow Ax \]

\[ \text{fold}_F = \lambda A.\lambda f.\lambda x.\lambda e. e A f \]

\[ \text{in}_F = \lambda x.\lambda e.\lambda A.\lambda f. f A (\text{fmap}_F (\mu F) A (\text{fold}_F A f) x e) \]

use relational parametricity to prove the \(\eta\)-law
Summary
Relationally parametric model of Dependent Types

{ Contexts as reflexive graphs
   Types as families of reflexive graphs

Applications of Dependently-Typed Parametricity

{ Free Theorems
  Initial Algebras for Indexed Types

Future work

{ Relationship with Homotopy Types?
  Higher Dimensions?
  Internalisation?
  Universe Hierarchy?
  Final coalgebras