Generalising Abstraction

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Off the Beaten Track
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What is Abstraction?

The Abstract is ‘an Enemy’: Alternative Perspectives to Computational Thinking

(Alan F. Blackwell, Luke Church, and Thomas Green, 2008)

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Abstract

An enemy.

1. Introduction

What is the worst possible name for a sc
software design class). Answer: “Proces

Abstraction: *categorisation?*

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LOCUS SOLUM: From the rules of logic to the logic of rules

(Jean-Yves Girard, 2001)

Appendix A: "A pure waste of paper"

• ABSTRACTION

Abstraction consists in treating things not as they are, but *as they should be*. Typically, when I replace a Michelin `XH1 TL 86H` with a `XH1 TL 86H`, I am not replacing a thing with its exact copy, I am just matching a specification. The new tire is different from the one it replaces, but it behaves in the same way, as long as I use it as a tire ; differences, like this small change of colour, do not matter. Observe that the specifications of tires are sufficiently precise so as to allow the replacement of only one of your two front tires with another of the same type considered as identical.

This applies to industry, which deals —say— not with food, but with the idea of food, think of Mc Donald's. Craft deals with objects as they are, no replacement is possible… for the best and for the worse.

Abstraction should not be confused with spiritualism, which is just abstraction from location.

See : **Implicit, Locative logic, Money, Pauperism, Specification, Spiritualism, Tradition**.

Programming with Abstract Data Types

(Barbara Liskov and Stephen Zilles, 1974)

This leads us to the concept of abstract data type which is central to the design of the language. An _abstract data type_ defines a class of abstract objects which is completely characterized by the operations available on those objects. This means that an abstract data type can be defined by defining the characterizing operations for that type.

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$$\begin{aligned} \mathsf{empty} &: A \\ \mathsf{insert} &: \mathsf{int} \to A \to A \\ \mathsf{member?} &: \mathsf{int} \to A \to \mathsf{bool} \end{aligned}$$

"operations available on those objects": Algebras?

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An idea!

- If clients only "know" about the operations
- and omomorphisms preserve operations
- ... are clients invariant under change through homomorphism?

Types, Abstraction, and Parametric Polymorphism
(John Reynolds, 1983)

We explore the thesis that type structure is a syntactic discipline for maintaining levels of abstraction. Traditionally, this view has been formalized algebraically, but the algebraic approach fails to encompass higher-order functions. For this purpose, it is necessary to generalize homomorphic functions to relations; the result is an "abstraction" theorem that is applicable to the typed lambda calculus and various extensions, including user-defined types.

Finally, we consider polymorphic functions, and show that the abstraction theorem captures Strachey's concept of parametric, as opposed to ad hoc, polymorphism.

Abstraction Theorem:

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$$\alpha_1, \ldots, \alpha_n; \Gamma \vdash M : A$$

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$$\alpha_1, \ldots, \alpha_n; \Gamma \vdash M : A$$

and related instantiations for the type variables

$$\begin{array}{cccc} X_1 & X_2 & \ldots & X_n \\ \updownarrow{\scriptstyle R_1} & \updownarrow{\scriptstyle R_2} & & \updownarrow{\scriptstyle R_n} \\ Y_1 & Y_2 & \ldots & Y_n \end{array}$$

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and environments: $\gamma_1 \in [\![\Gamma]\!](X_1, \ldots, X_n), \gamma_2 \in [\![\Gamma]\!](Y_1, \ldots, Y_n)$

$$(\gamma_1, \gamma_2) \in \mathcal{R}[\![\Gamma]\!](R_1, \ldots, R_n)$$

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and environments: $\gamma_1 \in [\![\Gamma]\!](X_1, \ldots, X_n), \gamma_2 \in [\![\Gamma]\!](Y_1, \ldots, Y_n)$

$$(\gamma_1, \gamma_2) \in \mathcal{R}[\![\Gamma]\!](R_1, \ldots, R_n)$$

then

$$([\![M]\!](X_1, \ldots, X_n)\gamma_1, [\![M]\!](Y_1, \ldots, Y_n)\gamma_2) \in \mathcal{R}[\![A]\!](R_1, \ldots, R_n)$$

Abstraction Theorem *(simplified)*:

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$$[\![M]\!]X\gamma_1 =_{\mathsf{bool}} [\![M]\!]Y\gamma_2$$

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$$[\![M]\!]X\gamma_1 =_{\mathsf{bool}} [\![M]\!]Y\gamma_2$$

The idea: If we switch X for Y, with sympathetic changes to the environment, then the result is unchanged.

Abstraction *itself* yields emergment properties

Without Loss of Generality

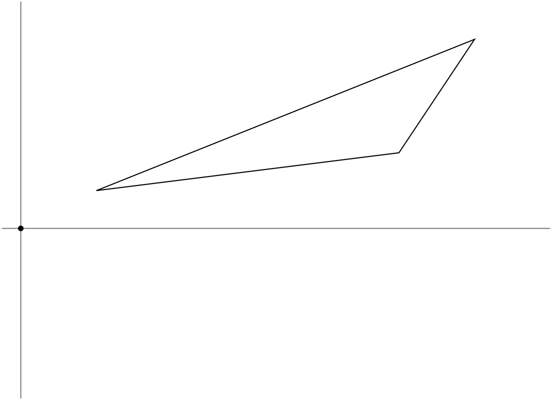
(John Harrison, 2009)

Abstract. One sometimes reads in a mathematical proof that a certain assumption can be made 'without loss of generality' (WLOG). In other words, it is claimed that considering what first appears only a special case does nevertheless suffice to prove the general result. Typically the intuitive justification for this is that one can exploit symmetry in the problem. We examine how to formalize such 'WLOG' arguments in a mechanical theorem prover. Geometric reasoning is particularly rich in examples and we pay special attention to this area.

Without Loss of Generality

(John Harrison, 2009)

For the purposes of computing *the area*:



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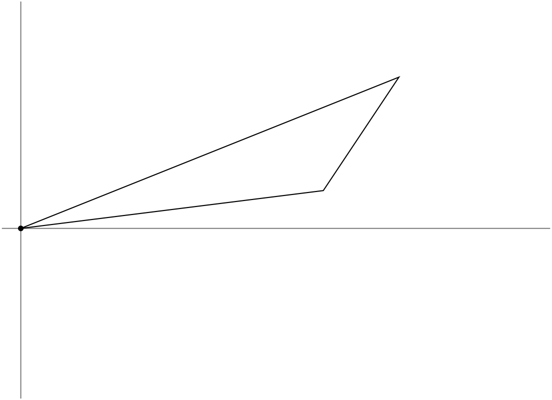

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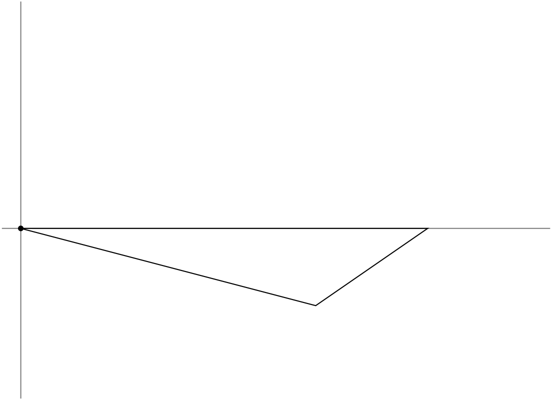

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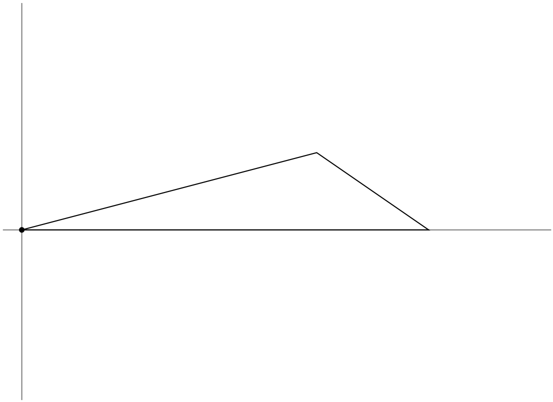

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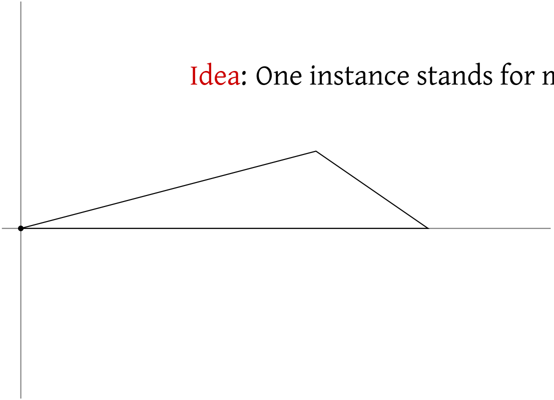

Idea: One instance stands for many

"*A rose by any other name would smell as sweet*"
(William Shakespeare, $\sim$1591–5)

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$$\lambda x.\, x \quad \equiv_\alpha \quad \lambda y.\, y \quad \equiv \quad \lambda .0$$

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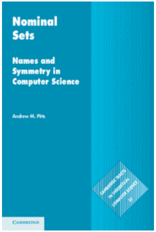
$$\lambda x.\, x \quad \equiv_\alpha \quad \lambda y.\, y \quad \equiv \quad \lambda.0$$

Nominal Sets: Names and Symmetry in Computer Science
(Andrew Pitts, 2013)



In contrast, *nominal techniques* provide a setting in which dependence of structures upon names is implicit. They achieve this by representing name independence, rather than dependence, in terms of invariance under symmetry, as I explain next.

from *Nominal Techniques*. "In contrast" to using semantic functions

For this talk:

Abstraction is explained by ***invariance under change***

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Abstraction is explained by ***invariance under change***

Change invariance yields ***interesting properties***

A Simple Model for Change Invariance

(Edmund Robinson and Guiseppe Rosolini, 1994)

(A., Neil Ghani, and Patricia Johann, 2014)

Type = objects + changes

Γ_O	:	Set	*set of objects*
Γ_R	:	$\Gamma_O \times \Gamma_O \to \mathsf{Set}$	*set of changes from one object to another*
Γ_{refl}	:	$\forall \gamma{:}\Gamma_O.\ \Gamma_R(\gamma, \gamma)$	*"do nothing" change*

Type = objects + changes

$$\Gamma_O : \mathsf{Set}$$ *set of objects*

$$\Gamma_R : \Gamma_O \times \Gamma_O \to \mathsf{Set}$$ *set of changes from one object to another*

$$\Gamma_{refl} : \forall \gamma{:}\Gamma_O.\ \Gamma_R(\gamma, \gamma)$$ *"do nothing" change*

Reflexive Graphs

1. Boring old sets

$$
\begin{aligned}
(\uparrow X)_O &= X \\
(\uparrow X)_R(x, y) &= \{* \mid x = y\} \\
(\uparrow X)_{refl}(x) &= *
\end{aligned}
$$

1. *Boring old sets* $(\uparrow X)_R(x, y) = \{* \mid x = y\}$.

2. Groups

$$
\begin{aligned}
(\mathsf{GL}_2)_O &= \{*\} \\
(\mathsf{GL}_2)_R(*, *) &= \textit{invertible linear transformations on } \mathbb{R}^2 \\
(\mathsf{GL}_2)_{\textit{refl}}(*) &= \mathrm{id}
\end{aligned}
$$

1. *Boring old sets* $(\uparrow X)_R(x, y) = \{* \mid x = y\}$.

2. *Groups* $G_R(*, *) = G$.

3. Groupoids

$$\begin{aligned} \mathsf{Perm}_o &= \mathbb{N} \\ \mathsf{Perm}_R(m, n) &= \{\pi \mid m = n, \pi \in S(n)\} \\ \mathsf{perm}_{\mathit{refl}}(n) &= \mathrm{id} \end{aligned}$$

1. *Boring old sets* $(\uparrow X)_R(x, y) = \{* \mid x = y\}$.

2. *Groups* $G_R(*, *) = G$.

3. *Groupoids* $G_R(x, y) = G(x, y)$

4. Distances

$$\begin{aligned} \mathsf{Dist}_O &= * \\ \mathsf{Dist}_R(*, *) &= \mathbb{R}^{\geq 0} \\ \mathsf{Dist}_{refl}(*) &= 0 \end{aligned}$$

1. *Boring old sets* $(\uparrow X)_R(x, y) = \{* \mid x = y\}$.
2. *Groups* $G_R(*, *) = G$.
3. *Groupoids* $G_R(x, y) = G(x, y)$
4. *Distances* $\mathsf{Dist}_R(*, *) = \mathbb{R}^{\geq 0}$
5. Small types

$$
\begin{aligned}
\mathsf{U}_O &= \mathsf{set} && \textit{(small sets)} \\
\mathsf{U}_R(X, Y) &= \mathrm{Rel}(X, Y) \\
\mathsf{U}_{\mathit{refl}}(X) &= \equiv_X
\end{aligned}
$$

1. Boring old sets $(\uparrow X)_R(x, y) = \{* \mid x = y\}$.
2. Groups $G_R(*, *) = G$.
3. Groupoids $G_R(x, y) = G(x, y)$
4. Distances $\mathsf{Dist}_R(*, *) = \mathbb{R}^{\geq 0}$
5. Small types $\mathsf{U}_R(X, Y) = \mathrm{Rel}(X, Y)$

Type families $A \in \mathrm{Ty}(\Gamma)$

$$
\begin{aligned}
A_O &: \Gamma_O \to \mathsf{Set} \\
A_R &: \forall \gamma_1, \gamma_2{:}\Gamma_O.\ \Gamma_R(\gamma_1, \gamma_2) \times A_O(\gamma_1) \times A_O(\gamma_2) \to \mathsf{Set} \\
A_{refl} &: \forall \gamma{:}\Gamma_O, a{:}A_O(\gamma).\ A_R(\gamma, \gamma, \Gamma_{refl}(\gamma), a, a)
\end{aligned}
$$

Numbers, under scaling

$$\begin{aligned}
\mathsf{num} &\in \mathsf{Ty}(\mathsf{GL}(1)) \\
\mathsf{num}_O(*) &= \mathbb{R} \\
\mathsf{num}_R(*, *, k, x, y) &= \{* \mid x = k \cdot y\} \\
\mathsf{num}_{refl}(*, x) &= *
\end{aligned}$$

Relational Parametricity and Units of Measure
(Andrew Kennedy, 1997)

Numbers, under scaling

$$\begin{aligned} \mathsf{num} &\in \mathsf{Ty}(\mathsf{GL}(1)) \\ \mathsf{num}_O(*) &= \mathbb{R} \\ \mathsf{num}_R(*, *, k, x, y) &= \{* \mid x = k \cdot y\} \\ \mathsf{num}_{\mathit{refl}}(*, x) &= * \end{aligned}$$

Relational Parametricity and Units of Measure
(Andrew Kennedy, 1997)

Numbers, under scaling

$$\begin{aligned} \mathsf{num} &\in \mathsf{Ty}(\mathsf{Dist}) \\ \mathsf{num}_O(*) &= \mathbb{R} \\ \mathsf{num}_R(*, *, d, x, y) &= \{* \mid |x - y| \leq d\} \\ \mathsf{num}_{\mathit{refl}}(*, x) &= * \end{aligned}$$

Distance Makes the Types Grow Stronger
(Jason Reed and Benjamin C. Pierce, 2010)

Decoding Small types

$$
\begin{aligned}
\mathsf{T} \quad & \in \mathrm{Ty}(\mathsf{U}) \\
\mathsf{T}_{O}(A) \quad & = A \\
\mathsf{T}_{R}(A, B, R, a, b) \quad & = \{* \mid R(a, b)\} \\
\mathsf{T}_{\mathit{refl}}(A, a) \quad & = *
\end{aligned}
$$

Terms $M \in \mathrm{Tm}(\Gamma, A)$ of parameterised type $A \in \mathrm{Ty}(\Gamma)$

$$\begin{aligned} M_o &: \forall \gamma{:}\Gamma_O.\ A_O(\gamma) \\ M_r &: \forall \gamma_1, \gamma_2{:}\Gamma_O, \gamma_r{:}\Gamma_R(\gamma_1, \gamma_2).\ A_R(\gamma_1, \gamma_2, \gamma_r, M_o(\gamma_1), M_o(\gamma_2)) \end{aligned}$$

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such that, for all $\gamma \in \Gamma_O$,

$$M_r(\gamma, \gamma, \Gamma_{\mathit{refl}}(\gamma)) = A_{\mathit{refl}}(\gamma, M_o(\gamma))$$

Terms $M \in \mathrm{Tm}(\Gamma, A)$ of parameterised type $A \in \mathrm{Ty}(\Gamma)$

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such that, for all $\gamma \in \Gamma_O$,

$$M_r(\gamma, \gamma, \Gamma_{refl}(\gamma)) = A_{refl}(\gamma, M_o(\gamma))$$

M_r generalises the Abstraction Theorem

Terms $M \in \mathrm{Tm}(\Gamma, A)$ of parameterised type $A \in \mathrm{Ty}(\Gamma)$

$$
\begin{aligned}
M_o &: \forall \gamma{:}\Gamma_O.\ A_O(\gamma) \\
M_r &: \forall \gamma_1, \gamma_2{:}\Gamma_O, \gamma_r{:}\Gamma_R(\gamma_1, \gamma_2).\ A_R(\gamma_1, \gamma_2, \gamma_r, M_o(\gamma_1), M_o(\gamma_2))
\end{aligned}
$$

such that, for all $\gamma \in \Gamma_O$,

$$
M_r(\gamma, \gamma, \Gamma_{refl}(\gamma)) = A_{refl}(\gamma, M_o(\gamma))
$$

M_r generalises the Abstraction Theorem

refl preservation generalises Identity Extension Lemma

Π-types: $\Pi AB \in \mathrm{Ty}(\Gamma)$ from $A \in \mathrm{Ty}(\Gamma), B \in \mathrm{Ty}(\Gamma.A)$

$$
\begin{aligned}
&(\Pi AB)_O(\gamma) \\
&\quad = \{(f, r) \mid f : \forall a \in A_O(\gamma).B_O(\gamma, a), \\
&\qquad\qquad r : \forall a_1, a_2 \in A_O(\gamma), a_r \in A_R(\gamma, \gamma, \Gamma_{refl}(\gamma))(a_1, a_2). \\
&\qquad\qquad\qquad B_R((\gamma, a_1), (\gamma, a_2), (\Gamma_{refl}(\gamma), a_r))(f(a_1), f(a_2)), \\
&\qquad\qquad \forall a.\ r(a, a, A_{refl}(\gamma, a)) = B_{refl}(\gamma, a)(f(a))\}
\end{aligned}
$$

$$
\begin{aligned}
&(\Pi AB)_R(\gamma_1, \gamma_2, \gamma_r)((f_1, r_1), (f_2, r_2)) \\
&\quad = \forall a_1 \in A_O(\gamma_1), a_2 \in A_O(\gamma_2), a_r \in A_R(\gamma_1, \gamma_2, \gamma_r)(a_1, a_2). \\
&\qquad\qquad B_R((\gamma_1, a_1), (\gamma_2, a_2), (\gamma_r, a_r))(f_1(a_1), f_2(a_2))
\end{aligned}
$$

Only the *change invariant* functions.

Using the Model

"Free" Theorems

(Philip Wadler, 1989)

$$M : \Pi\alpha{:}\mathsf{U}.\ \mathsf{T}\alpha \to \mathsf{T}\alpha$$

interpreted as:

$M_o\ :\ \forall X{:}\mathsf{set}.\ X \to X$

$M_r\ :\ \forall X, Y{:}\mathsf{set}, R{:}\mathsf{Rel}(X, Y), x{:}X, y{:}Y. (x, y) \in R \Rightarrow (M_o\ X\ x, M_o\ Y\ y) \in R$

Special case: for all X and x, $M_o\ X\ x = x$.

Geometric Group Indexed Types

Parameterised Types for Geometry:

$$\begin{array}{lcl} \mathbb{R}^n & : & \mathsf{GL}(n) \to \mathsf{T}(n) \to \mathsf{CartSp} \\ C^\infty & : & \mathsf{CartSp} \to \mathsf{CartSp} \to \mathsf{U} \\ \wr - \int & : & \mathsf{CartSp} \to \mathsf{U} \end{array}$$

where

$\mathsf{GL}(n)$	–	invertible linear transformations
$\mathsf{O}(n)$	–	orthogonal linear transformations
$\mathsf{T}(n)$	–	translations
CartSp	–	cartesian spaces

$$\mathit{area} : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).$$
$$\qquad \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle|\det B|, 0\rangle$$

$$area : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).\\ \qquad \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle |\det B|, 0\rangle$$

Symmetry as a Guide:

$$\mathit{area} : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).\ \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle |\det B|, 0\rangle$$

Symmetry as a Guide:

We have $p_1 : \mathbb{R}^2\langle B, t\rangle, p_2 : \mathbb{R}^2\langle B, t\rangle, p_3 : \mathbb{R}^2\langle B, t\rangle$

$$area : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).$$
$$\mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle|\det B|, 0\rangle$$

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goal is $\mathbb{R}\langle |\det B|, 0\rangle$

select one to be the "origin":

$(p_2 - p_1) : \mathbb{R}^2\langle B, 0\rangle$ and $(p_3 - p_1) : \mathbb{R}^2\langle B, 0\rangle$

$$area : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).$$
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select one to be the "origin":

$(p_2 - p_1) : \mathbb{R}^2\langle B, 0\rangle$ and $(p_3 - p_1) : \mathbb{R}^2\langle B, 0\rangle$

remove rotational symmetry and get area of parallelogram:

$(p_2 - p_1) \times (p_3 - p_1) : \mathbb{R}\langle \det B, 0\rangle$

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remove rotational symmetry and get area of parallelogram:

$(p_2 - p_1) \times (p_3 - p_1) : \mathbb{R}\langle \det B, 0\rangle$

remove reflectional symmetry:

$|(p_2 - p_1) \times (p_3 - p_1)| : \mathbb{R}\langle |\det B|, 0\rangle$

$$area : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2). \quad \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle |\det B|, 0\rangle$$

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$(p_2 - p_1) \times (p_3 - p_1) : \mathbb{R}\langle \det B, 0\rangle$

remove reflectional symmetry:

$|(p_2 - p_1) \times (p_3 - p_1)| : \mathbb{R}\langle |\det B|, 0\rangle$

halve:

$\frac{1}{2} * |(p_2 - p_1) \times (p_3 - p_1)| : \mathbb{R}\langle |\det B|, 0\rangle$

$$area : \Pi B{:}\mathsf{GL}(2), t{:}\mathsf{T}(2).$$
$$\mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}^2\langle B, t\rangle \to \mathbb{R}\langle |\det B|, 0\rangle$$

Symmetry as a Guide:

We have $p_1 : \mathbb{R}^2\langle B, t\rangle, p_2 : \mathbb{R}^2\langle B, t\rangle, p_3 : \mathbb{R}^2\langle B, t\rangle$

goal is $\mathbb{R}\langle |\det B|, 0\rangle$

select one to be the "origin":

$(p_2 - p_1) : \mathbb{R}^2\langle B, 0\rangle$ and $(p_3 - p_1) : \mathbb{R}^2\langle B, 0\rangle$

remove rotational symmetry and get area of parallelogram:

$(p_2 - p_1) \times (p_3 - p_1) : \mathbb{R}\langle \det B, 0\rangle$

remove reflectional symmetry:

$|(p_2 - p_1) \times (p_3 - p_1)| : \mathbb{R}\langle |\det B|, 0\rangle$

halve:

$\frac{1}{2} * |(p_2 - p_1) \times (p_3 - p_1)| : \mathbb{R}\langle |\det B|, 0\rangle$

Each step is a type isomorphism, justified by change invariance.

Term constants for Classical Mechanics:

$$\underline{\vec{c}} \quad : \ \wr\, \mathbb{R}^n\langle 1, 0\rangle \int$$

$$0 \quad : \Pi g{:}\mathsf{GL}(n).\ \wr\, \mathbb{R}^n\langle g, 0\rangle \int$$

$$(+) \quad : \Pi g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 + t_2\rangle)$$

$$(-) \quad : \Pi g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 - t_2\rangle)$$

$$(*) \quad : \Pi g_1{:}\mathsf{GL}(1), g_2{:}\mathsf{GL}(n).$$
$$C^\infty(\mathbb{R}\langle g_1, 0\rangle \times \mathbb{R}^n\langle g_2, 0\rangle, \mathbb{R}\langle \mathrm{scale}_n(g_1) g_2, 0\rangle)$$

$$(\cdot) \quad : \Pi g{:}\mathsf{GL}(1), o{:}\mathsf{O}(n).$$
$$C^\infty(\mathbb{R}^n\langle(\mathrm{scale}_n\, g)(\mathrm{ortho}_n\, o), 0\rangle \times$$
$$\mathbb{R}^n\langle(\mathrm{scale}_n\, g)(\mathrm{ortho}_n\, o), 0\rangle, \mathbb{R}\langle(\mathrm{scale}_n\, g)^2, 0\rangle)$$

Term constants for Classical Mechanics:

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$$(-) \quad : \Pi g{:}\mathsf{GL}(n), t_1, t_2{:}\mathsf{T}(n).\ C^\infty(\mathbb{R}^n\langle g, t_1\rangle \times \mathbb{R}^n\langle g, t_2\rangle, \mathbb{R}^n\langle g, t_1 - t_2\rangle)$$

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$$(\cdot) \quad : \Pi g{:}\mathsf{GL}(1), o{:}\mathsf{O}(n).\ C^\infty(\mathbb{R}^n\langle(\mathsf{scale}_n\, g)(\mathsf{ortho}_n\, o), 0\rangle \times \mathbb{R}^n\langle(\mathsf{scale}_n\, g)(\mathsf{ortho}_n\, o), 0\rangle, \mathbb{R}\langle(\mathsf{scale}_n\, g)^2, 0\rangle)$$

Term constants for Classical Mechanics:

$$\underline{\vec{c}} \quad : \wr \mathbb{R}^n\langle 1, 0\rangle \int$$

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$$(\cdot) \quad : \Pi g{:}\mathsf{GL}(1), o{:}\mathsf{O}(n).\ C^\infty(\mathbb{R}^n\langle(\mathrm{scale}_n\, g)(\mathrm{ortho}_n\, o), 0\rangle \times \mathbb{R}^n\langle(\mathrm{scale}_n\, g)(\mathrm{ortho}_n\, o), 0\rangle, \mathbb{R}\langle(\mathrm{scale}_n\, g)^2, 0\rangle)$$

$$
\begin{aligned}
\mathrm{sin} &: \Pi z{:}\mathsf{Z}.\ C^{\infty}(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\mathrm{cos} &: \Pi z{:}\mathsf{Z}.\ C^{\infty}(\mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \mathbb{R}\langle 1, 0\rangle) \\
\mathrm{exp} &: \Pi t{:}\mathsf{T}(1).\ C^{\infty}(\mathbb{R}\langle 1, t\rangle, \mathbb{R}\langle \exp t, 0\rangle) \\
\mathrm{const} &: \wr Y \int \to C^{\infty}(X, Y) \\
\mathrm{id} &: C^{\infty}(X, X) \\
(\ggg) &: C^{\infty}(X, Y) \to C^{\infty}(Y, Z) \to C^{\infty}(X, Z)
\end{aligned}
$$

Free Particle

$$\Gamma = \; t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3), m : \wr \mathbb{R}\langle 1, 0\rangle \int$$

$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), t_x\rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle$$

$$L = \frac{1}{2} m(\dot{x} \cdot \dot{x}) : \mathbb{R}\langle 1, 0\rangle$$

Free Particle

$$\Gamma = t_t : \mathsf{T}(1), t_x : \mathsf{T}(3), o : \mathsf{O}(3), m : \wr\mathbb{R}\langle 1, 0\rangle\int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, x : \mathbb{R}^3\langle \mathrm{ortho}_3(o), t_x\rangle, \dot{x} : \mathbb{R}^3\langle \mathrm{ortho}_3(o), 0\rangle$$

$$L = \frac{1}{2}m(\dot{x}\cdot\dot{x}) : \mathbb{R}\langle 1, 0\rangle$$

Free theorems ⇒ Conservation laws

(Emmy Noether, 1918)

$\forall t_t \in \mathbb{R}.\ [\![L]\!](t + t_t, \vec{x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	energy
$\forall \vec{t_x} \in \mathbb{R}^3.\ [\![L]\!](t, \vec{x} + \vec{t_x}, \dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	linear momentum
$\forall O \in \mathsf{O}(3).\ [\![L]\!](t, O\vec{x}, O\dot{\vec{x}}) = [\![L]\!](t, \vec{x}, \dot{\vec{x}})$	$\Rightarrow$	angular momentum

Pendulum:

$$\Gamma = \ t_t : \mathsf{T}(1), z : \mathsf{Z}, m : \wr \mathbb{R}\langle 1, 0\rangle \int, l : \wr \mathbb{R}\langle 1, 0\rangle \int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle$$

Pendulum:

$$\Gamma = t_t : \mathsf{T}(1), z : \mathsf{Z}, m : \wr\mathbb{R}\langle 1, 0\rangle\int, l : \wr\mathbb{R}\langle 1, 0\rangle\int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle$$

$$\begin{aligned} L = {} & \text{let } y = l \sin\theta \text{ in} \\ & \text{let } \dot{x} = l\dot{\theta}\cos\theta \text{ in} \\ & \text{let } \dot{y} = -l\dot{\theta}\sin\theta \text{ in} \\ & \quad \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy : \mathbb{R}\langle 1, 0\rangle \end{aligned}$$

Pendulum:

$$\Gamma = t_t : \mathsf{T}(1), z : \mathsf{Z}, m : \wr\mathbb{R}\langle 1, 0\rangle\int, l : \wr\mathbb{R}\langle 1, 0\rangle\int$$
$$\Delta = t : \mathbb{R}\langle 1, t_t\rangle, \theta : \mathbb{R}\langle 1, \underline{2\pi} * z\rangle, \dot{\theta} : \mathbb{R}\langle 1, 0\rangle$$

$$\begin{aligned} L = {} & \text{let } y = l \sin\theta \text{ in} \\ & \text{let } \dot{x} = l\dot{\theta}\cos\theta \text{ in} \\ & \text{let } \dot{y} = -l\dot{\theta}\sin\theta \text{ in} \\ & \quad \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy : \mathbb{R}\langle 1, 0\rangle \end{aligned}$$

Free theorems:

Energy conservation

Invariance under $z : \mathsf{Z}$ not smooth $\Rightarrow$ no conserved property

Non-example

Higher-Order Abstract Syntax

$$\forall\alpha.\begin{pmatrix}\textit{lam:} & ((\alpha \to \alpha) \to \alpha) \to \\ \textit{app:} & (\alpha \to \alpha \to \alpha) \to \\ & \alpha\end{pmatrix} \cong \text{closed } \lambda\text{-terms}$$

Seems to need *Kripke-indexed* relations.

de-Meta-ing the Model

The *Martin-Löf Identity Type*:

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash a_1 : A \qquad \Gamma \vdash a_2 : A}{\Gamma \vdash \mathsf{Id}(A, a_1, a_2) \ \mathsf{type}}$$

The *Martin-Löf Identity Type*:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a_1 : A \qquad \Gamma \vdash a_2 : A}{\Gamma \vdash \mathsf{Id}(A, a_1, a_2) \text{ type}}$$

So why not a *Relatedness Type*:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash a_1 : A \qquad \Gamma \vdash a_2 : A}{\Gamma \vdash \mathsf{Rel}(A, a_1, a_2) \text{ type}}$$

Objects are *equal to themselves*:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}(a) : \mathsf{Id}(A, a, a)}$$

Objects are *equal to themselves*:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}(a) : \mathsf{Id}(A, a, a)}$$

Objects are *related to themselves*:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}(a) : \mathsf{Rel}(A, a, a)}$$

Equality *reflects*:

$$\frac{\Gamma \vdash P : \mathsf{Id}(A, a_1, a_2)}{\Gamma \vdash a_1 \equiv a_2 : A} \qquad \frac{\Gamma \vdash P : \mathsf{Id}(A, a_1, a_2)}{\Gamma \vdash P \equiv \mathsf{refl}(a_1) : \mathsf{Id}(A, a_1, a_2)}$$

Equality *reflects*:

$$\frac{\Gamma \vdash P : \mathsf{Id}(A, a_1, a_2)}{\Gamma \vdash a_1 \equiv a_2 : A} \qquad \frac{\Gamma \vdash P : \mathsf{Id}(A, a_1, a_2)}{\Gamma \vdash P \equiv \mathsf{refl}(a_1) : \mathsf{Id}(A, a_1, a_2)}$$

Relatedness *computes*:

$$\begin{aligned}
\mathrm{Rel}(\rho, \mathsf{U}, M_1, M_2) &\equiv \mathsf{T}M_1 \to \mathsf{T}M_2 \to \mathsf{Prop} \\
\mathrm{Rel}(\rho, \Pi x{:}A.B, M_1, M_2) &\equiv \Pi x_1{:}A\{\mathsf{src}(\rho)\}. \\
&\qquad \Pi x_2{:}A\{\mathsf{tgt}(\rho)\}. \\
&\qquad\quad \Pi x_r{:}\mathrm{Rel}(\rho, A, x_1, x_2). \\
&\qquad\qquad \mathrm{Rel}((\rho, x = \langle x_1, x_2, x_r\rangle), B, M_1 x_1, M_2 x_2) \\
\mathrm{Rel}(\rho, \mathsf{T}X, M_1, M_2) &\equiv \mathsf{T}(\mathrm{refl}(\rho, X)\ M_1\ M_2)
\end{aligned}$$

$$\begin{aligned}
\mathrm{refl}(\rho, x) &\equiv \rho(x) \\
\mathrm{refl}(\rho, MN) &\equiv \mathrm{refl}(\rho, M)\ N\{\mathsf{src}(\rho)\}\ N\{\mathsf{tgt}(\rho)\}\ \mathrm{refl}(\rho, N) \\
\mathrm{refl}(\rho, \lambda x.M) &\equiv \lambda x_1.\lambda x_2.\lambda x_r.\mathrm{refl}((\rho, x = \langle x_1, x_2, x_r\rangle), M)
\end{aligned}$$

Link back to *equality*:

$$\frac{\Gamma \vdash P : \mathsf{Prf}(\mathrm{refl}(X)\ M_1\ M_2)}{\Gamma \vdash M_1 \equiv M_2 : \mathsf{T}X}$$

$$\frac{\Gamma \vdash P : \mathsf{Prf}(\mathrm{refl}(X)\ M_1\ M_2)}{\Gamma \vdash P \equiv \mathrm{refl}(M_1) : \mathsf{Prf}(\mathrm{refl}(X)\ M_1\ M_2)}$$

Internalises: reflexive relation on small types is equality.

Future work...

WLOG Programming?

Types for Scientific Computing?

Types for Scientific Theories?

Constructive Change for Incremental Computing?

Making Abstraction Visible?

Was that a good model of Change Invariance?