Resource Constrained Programming with Full Dependent Types

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Goal of this talk —

– To extend Implicit Computational Complexity to Dependent Types
– Motivations:
  – Certified resource constrained programming
  – Resource constrained mathematics?
– Using *Quantitative Type Theory*. 
Programming in Type Theory
Programming in Type Theory —

\[ \vdash M : \text{Vec Nat} \ 5 \]

\[ \vdash \text{check} : (t : \text{Tm}) \rightarrow (a : \text{Ty}) \rightarrow (\vdash' t \ a) + \lnot(\vdash' t \ a) \]

- An expressive typed functional language
- Implemented in Agda, Coq, Idris, Lean, ...
Programming in Type Theory —

\[\textbf{data } \text{Vec} \ (A : \text{Set}) : \text{Nat} \rightarrow \text{Set} \ \textbf{where}\]
\[\text{nil} : \text{Vec} \ A \ 0\]
\[\text{cons} : (n : \text{Nat}) \rightarrow A \rightarrow \text{Vec} \ A \ n \rightarrow \text{Vec} \ A \ (S \ n)\]

— Standard compilation scheme —
— cons constructor has three arguments
— Space consumption is \(O(n^2)\)!
Type Theory is tough to compile —

- Extraneous data must be erased
e.g., sizes in vectors

- The compilation target is assumed to have unlimited capacity —
  - In time and space
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- Extraneous data must be erased
e.g., sizes in vectors

- The compilation target is assumed to have unlimited capacity —
  - In time and space

Constrain the resources required —

- Use *Quantitative Type Theory* to control resource usage
- Use Implicit Computational Complexity ideas to restrict time complexity
Two Systems by Martin Hofmann
Implicit Computational Complexity —

- Capturing complexity classes via restricted functional languages
- Closely related to static analysis for resource consumption
- Going to look at two systems by Martin Hofmann:
  - LFPL "Linear Function Programming Language"
  - Amortised Resource Analysis
- Controlling (nested) iteration is the key.
Warm-up: Linear time system —

- In the linear λ-calculus
- A natural number iterator:

\[
\begin{align*}
\Gamma_1 \vdash M_z &: A \\
\Gamma_1 &\vdash x &: A \\
\Gamma_2 &\vdash N &: \text{Nat} \\
\Gamma_1, \Gamma_2 &\vdash \text{iter}_A(M_z, x.M_s, N) &: A
\end{align*}
\]

- Once we iterate over some number \( N \) it is *used up*
Warm-up: Linear time system —

— In the linear $\lambda$-calculus

— A natural number iterator:

$$
\frac{
\Gamma_1 \vdash M_z : A \quad x : A \vdash M_s : A \quad \Gamma_2 \vdash N : \text{Nat}
}{
\Gamma_1, \Gamma_2 \vdash \text{iter}_A(M_z, x.M_s, N) : A
}$$

— Once we iterate over some number $N$ it is used up

— No way to construct new “iterable” natural numbers
  — If we could, then could easily duplicate Nats
  — then iterate to get multiplication
  — and again to get exponentiation

— Can have other constructable data that is not iterable: Nat°
LFPL: extending to polynomial time

- Introduce a type $\diamond$, representing a chunk of iterability
- Building a Nat requires $\diamond$s:
  
  \[
  0 : \diamond \rightarrow \text{Nat} \quad \text{S : } \diamond \rightarrow \text{Nat} \rightarrow \text{Nat}
  \]

- Iteration gives you them back:
  
  \[
  \Gamma_1, d : \diamond \vdash M_z : A \quad x : A, d : \diamond \vdash M_s : A \quad \Gamma_2 \vdash N : \text{Nat} \\
  \Gamma_1, \Gamma_2 \vdash \text{iter}_A(d.M_z, x \cdot d.M_s, N) : A
  \]

- Conservation of iterability
Iterating a function —

Assume we have a function $f: A \rightarrow A$
e.g., one step of a Turing machine

Linear $\binom{n}{1}$ iterations:

$$\text{iter}(\cdots) : \text{Nat} \otimes A \rightarrow \text{Nat} \otimes A$$

Reconstruct the Nat on the way through

$(\binom{n}{2})$ iterations:

$$\text{iter}(\cdots) : \text{Nat} \otimes A \rightarrow \text{Nat} \otimes A$$

Reconstruct the Nat on the way through
Do a nested linear iteration on the reconstructed number

$(\binom{n}{3})$ iterations: Iterate the above
Iterating a function —

- Obtain a \( \binom{n}{k} \) iterator for any \( k \)
- And get the original number back!
- Chain them together to get any polynomial:

\[
p(n) = \sum_{i=0}^{k} p_i \binom{n}{k}
\]

- So we get polytime completeness
Amortised Resource Analysis — (Hofmann & Jost, POPL 2003)

- Reinterpret ◊ as the cost of a step of iteration
- Inspired by Tarjan’s *amortised complexity analysis*
  - storing potential inside data structures
- Building a Nat still requires ◊s:

  \[
  0 : ◊ \to \text{Nat} \qquad S : ◊ \to \text{Nat} \to \text{Nat}
  \]

- But iteration no longer gives you them back:

  \[
  \Gamma_1 \vdash M_z : A \quad x : A \vdash M_s : A \quad \Gamma_2 \vdash N : \text{Nat} \Rightarrow \Gamma_1, \Gamma_2 \vdash \text{iter}_A(M_z, x.M_s, N) : A
  \]

- Back to linear time...
More flexibility

- Annotate data structures with number of $\diamond$s per constructor
  $\text{Nat}^p$

- Duplication:
  $\text{Nat}^{p_1+p_2} \to \text{Nat}^{p_1} \otimes \text{Nat}^{p_2}$

- Hofmann & Jost used linear programming to infer the $p$s
Regaining polynomial time — (Hoffmann & Hofmann, ESOP 2010)

— Annotate with sequences of naturals:

\[ \text{Nat}(p_1, \ldots, p_k) \]

— Interpretation is that

\[ \sum_{i=1}^{k} p_i \binom{n}{i} \]

number of ◇s is attached to a natural \( n \).
Annotate with sequences of naturals:

\[ \text{Nat}^{(p_1, \ldots, p_k)} \]

Interpretation is that

\[ \sum_{i=1}^{k} p_i \binom{n}{i} \]

number of \( \Diamond \)s is attached to a natural \( n \).

Iterator:

\[
\Gamma_1 \vdash M_z : A \quad n : \text{Nat}^{(p_1 + p_2, p_2 + p_3, \ldots, p_k)} \quad d : \Diamond p_1 \quad x : A \vdash M_s : A \quad \Gamma_2 \vdash N : \text{Nat}^{(p_1 + 1, \ldots, p_k)}
\]

\[
\Gamma_1, \Gamma_2 \vdash \text{iter}(M_z, n \; d \; x. M_s, N) : A
\]
Adapting these systems to dependent types
Dependency and Accountancy
In Martin-Löf Type Theory

\[ x_1 : S_1, \ldots, x_n : S_n \vdash M : T \]
In Martin-Löf Type Theory

\[ x_1 : S_1, \ldots, x_n : S_n \vdash M : T \]

variables \( x_1, \ldots, x_n \) are mixed usage
\[ n : Nat, x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]
\[ n : \text{Nat}, x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

\( x \) is used \textit{computationally}
\[ n : \text{Nat}, x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

\( x \) is used \textit{computationally}

\( n \) is used \textit{logically}
In Linear Logic

\[ x_1 : X_1, \ldots, x_n : X_n \vdash M : Y \]
In Linear Logic

\[ x_1 : X_1, \ldots, x_n : X_n \vdash M : Y \]

the presence of a variable \( x \) records its usage
each \( x_i \) must be “used” by \( M \) exactly once
In Linear Logic

$$x_1 : X_1, \ldots, x_n : X_n \vdash M : Y$$

the presence of a variable $x$ records its usage
each $x_i$ must be “used” by $M$ exactly once

Enables:

1. Insight into computational behaviour
2. e.g., time complexity
$n : \text{Nat}, x : \text{Fin}(n) \vdash x : \text{Fin}(n)$

Can we read this judgement linearly?
\[ n : \text{Nat}, x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

Can we read this judgement linearly?

\( n \) appears in the context, but is not used computationally
Can we read this judgement linearly?

▷ $n$ appears in the context, but is not used computationally

▷ $n$ appears twice in types
\[ n : \text{Nat}, x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

Can we read this judgement linearly?

▷ \( n \) appears in the context, but is not used computationally

▷ \( n \) appears \textit{twice} in types

Is \( n \) even used at all?
\[ n : \text{Nat} \mid x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]
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▷ Separate **intuitionistic / unrestricted** uses and **linear** uses
\[ n : \text{Nat} \mid x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

▷ Separate *intuitionistic / unrestricted* uses and *linear* uses

▷ Types can depend on intuitionistic data, but not linear data

*will come back to this...*
\[ n : \text{Nat} \mid x : \text{Fin}(n) \vdash x : \text{Fin}(n) \]

- Separate *intuitionistic / unrestricted* uses and *linear* uses

- Types can depend on intuitionistic data, but not linear data

(Barber, 1996)
(Cervesato and Pfenning, 2002)
(Krishnaswami, Pradic, and Benton, 2015)
(Vákár, 2015)
Separation interferes with dependency:

\[ n \colon \text{Nat} \mid x \colon \text{Fin}(n) \vdash (x, \text{refl}(x)) : (y \colon \text{Fin}(n)) \times (x \equiv y) \]
Separation interferes with dependency:

\[ n : \text{Nat} \mid x : \text{Fin}(n) \vdash (x, \text{refl}(x)) : (y : \text{Fin}(n)) \times (x \equiv y) \]

\[ n : \text{Nat}, x : \text{Fin}(n) \mid \hat{x} : \hat{\text{Fin}}(n, x) \vdash (x, \hat{x}, \text{refl}(x)) : (y : \text{Fin}(n)) \times \hat{\text{Fin}}(n, y) \otimes (x \equiv y) \]
Quantitative Coeffect calculi:

\[ x_1^{\rho_1}, \ldots, x_n^{\rho_n} : S_n \vdash M : T \]
Quantitative Coeffect calculi:

\[ x_1^{\rho_1}, \ldots, x_n^{\rho_n} : S \vdash M : T \]

- The \( \rho_i \) record usage from some semiring \( R \)
  - \( 1 \in R \) — a use
  - \( 0 \in R \) — not used
  - \( \rho_1 + \rho_2 \) — adding up uses (e.g., in an application)
  - \( \rho_1\rho_2 \) — nested uses
Quantitative Coeffect calculi:

\[ x_1^{\rho_1}, \ldots, x_n^{\rho_n} : S \vdash M : T \]

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  - \( 0 \in R \) — not used
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  - \( \rho_1 \rho_2 \) — nested uses

(Petricek, Orchard, and Mycroft, 2014)
(Brunel, Gaboardi, Mazza, and Zdancewic, 2014)
(Ghica and Smith, 2014)
Can we adapt this idea to dependent types?
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McBride’s idea:
▷ allow 0-usage data to appear in types.
   (McBride, 2016)
Can we adapt this idea to dependent types?

McBride’s idea:
▷ allow 0-usage data to appear in types.

(McBride, 2016)

\[ x_1^{\rho_1} S_1, \ldots, x_n^{\rho_n} S_n \vdash M^\sigma : T \]

where \( \sigma \in \{0, 1\} \).
▷ \( \sigma = 1 \) — the “real” computational world
▷ \( \sigma = 0 \) — the types world

(allowing arbitrary \( \rho \) yields a system where substitution is inadmissible)
Can we adapt this idea to dependent types?

McBride’s idea:
▷ allow 0-usage data to appear in types. 
  
(McBride, 2016)

\[ x_1^{\rho_1} : S_1, \ldots, x_n^{\rho_n} : S_n \vdash M^{\sigma} : T \]

where \( \sigma \in \{0, 1\} \).
  
▷ \( \sigma = 1 \) — the “real” computational world
▷ \( \sigma = 0 \) — the types world

(allowing arbitrary \( \rho \) yields a system where substitution is inadmissible)

Zero-ing is an admissible rule: 
\[
\frac{\Gamma \vdash M^1 : T}{0\Gamma \vdash M^0 : T}
\] allowing promotion to the type world.
Quantitative Type Theory
Quantitative Type Theory

Contexts

\[
\begin{align*}
\Gamma & \vdash 0 \Gamma \vdash S \\
\Gamma, x : S & \vdash \text{Ext}
\end{align*}
\]
Quantitative Type Theory

Contexts

\[

d \vdash \quad \text{EMP} \quad \vdash
\]

Types

\[
0 \Gamma \vdash S
\]
Quantitative Type Theory

**Contexts**

\[ \diamond \vdash \]

\[ \text{EMP} \]

\[ \Gamma \vdash 0 \Gamma \vdash S \]

\[ \text{Ext} \]

\[ \Gamma, x^{\rho} : S \vdash \]

**Types**

\[ 0 \Gamma \vdash S \]

**Terms**

\[ 0 \Gamma, x^{\sigma} : S, 0 \Gamma' \vdash \]

\[ \text{VAR} \]

\[ 0 \Gamma, x^{\sigma} : S, 0 \Gamma' \vdash x^{\sigma} : S \]

\[ \Gamma \vdash M^{\sigma} : S \]

\[ 0 \Gamma \vdash S \equiv T \]

\[ \text{Conv} \]

\[ \Gamma \vdash M^{\sigma} : T \]
\[
\begin{align*}
\Pi\text{-type formation} \\
0\Gamma \vdash S & \quad 0\Gamma, x^0 \vdash T \\
\hline
0\Gamma \vdash (x^\pi : S) \rightarrow T
\end{align*}
\]
Quantitative Type Theory

**Π-type formation**

\[
\begin{array}{c}
\frac{}{0\Gamma \vdash S} \quad \frac{0\Gamma, x^0 \vdash T}{0\Gamma \vdash (x^\pi : S) \rightarrow T}
\end{array}
\]

**Π-type introduction and elimination**

\[
\begin{array}{c}
\frac{\Gamma, x^{\sigma \pi} \vdash S \vdash M^\sigma \vdash T}{\Gamma \vdash \lambda x^{\pi} : S. M^\sigma : (x^\pi : S) \rightarrow T}
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma_1 \vdash M^\sigma : (x^\pi : S) \rightarrow T \quad \Gamma_2 \vdash N^{\sigma'} \vdash S \quad 0\Gamma_1 = 0\Gamma_2 \quad \sigma' = 0 \iff (\pi = 0 \lor \sigma = 0)}{\Gamma_1 + \pi\Gamma_2 \vdash MN^{\sigma} : T[N/x]}
\end{array}
\]
Integrating Resource Constraints into QTT
A suitable semiring for affine linearity?

- Carrier: \(\{0, 1, \omega\}\)
- Ordered: \(\omega < 1 < 0\)
- Operations:

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Would admit an unrestricted \(!\) modality.
Strict resource counting

- Carrier: $\mathbb{N}$
- Ordered: $\cdots < 2 < 1 < 0$
- Operations: normal operations on $\mathbb{N}$
Diamonds —

\[
\frac{\Gamma \vdash T_{Y-DIA}}{0\Gamma \vdash \Diamond} \quad \frac{\quad \frac{0\Gamma \vdash 0}{0\Gamma \vdash \ast : \Diamond}}{\quad T_{M-DIA}}
\]

In the $\sigma = 0$ fragment, $\Diamond$s are free.
- Natural number introduction

\[
\Gamma \vdash d^\sigma \quad \diamond \\
\Gamma \vdash \text{zero} @ d^\sigma : \text{Nat} \quad \Gamma \vdash n^\sigma : \text{Nat} \\
\Gamma \vdash \text{succ}(n) @ d^\sigma : \text{Nat}
\]

- Natural number elimination (\(\sigma = 1\) case)

\[
\begin{align*}
0 & : \Gamma, x : \text{Nat} \vdash A \\
\Gamma_1, d^1 & \vdash M_z^1 : A\{\text{zero}@ * / x}\end{align*}
\]
\[
\begin{align*}
d^1 & \vdash \diamond, n^0 : \text{Nat}, r^1 : A\{n/x\} \vdash M_s^1 : A\{\text{succ}(n)@ * / x\} \\
\Gamma_2 & \vdash N^1 : \text{Nat} \quad \Gamma_1 + \Gamma_2 = \Gamma\end{align*}
\]
\[
\Gamma \vdash \text{iter}(x.A, d.M_z, d\ n\ r.M_s, N) : A\{N/x\}
\]

- Crucial: \(n\) is not available for computational use in \(M_s\).
Encoding lists

- Define (in $\sigma = 0$ fragment):

$$\text{Vec } A : \text{Nat} \rightarrow \text{Set}$$

by iteration on the natural number.

- Lists:

$$\text{List } A = (n^1 : \text{Nat}) \otimes \text{Vec } A \ n$$
Amortised Analysis

- Unrestricted introduction rules for natural numbers:

\[
\begin{align*}
\Gamma & \vdash \Gamma \vdash \text{zero}^\sigma : \text{Nat} \\
\Gamma & \vdash \Gamma \vdash \text{succ}(N)^\sigma : \text{Nat}
\end{align*}
\]

- Postulate:

\[\Diamond^{(p_1, \ldots, p_k)} : \text{Nat} \to \text{Set}\]

- with:

\[\text{split} : (n^0 : \text{Nat}) \to \Diamond^{(p_1+p'_1, \ldots, p_k+p'_k)}(n) \to \Diamond^{(p_1, \ldots, p_k)}(n) \otimes \Diamond^{(p'_1, \ldots, p'_k)}(n)\]

\[\text{join} : (n^0 : \text{Nat}) \to \Diamond^{(p_1, \ldots, p_k)}(n) \otimes \Diamond^{(p'_1, \ldots, p'_k)}(n) \to \Diamond^{(p_1+p'_1, \ldots, p_k+p'_k)}(n)\]

\[\text{shift} : (n^0 : \text{Nat}) \to \Diamond^{(p_1, \ldots, p_k)}(\text{succ}(n)) \to \Diamond^{(p_1+p_2, \ldots, p_k)}(n)\]
Amortised Analysis

— Natural number elimination ($\sigma = 1$ case)

\[
\begin{align*}
0\Gamma, x : \text{Nat} &\vdash A \\
\Gamma_1 &\vdash M_z : A\{\text{zero}/x\} \\
n : \text{Nat}, r : A\{n/x\} &\vdash M_s : A\{\text{succ}(n)/x\} \\
\Gamma_2 &\vdash N : \text{Nat} \\
\Gamma_3 &\vdash D : \Diamond^{(1)}(N) \\
\Gamma_1 + \Gamma_2 + \Gamma_3 & = \Gamma \\
\Gamma &\vdash \text{iter}(x.A, M_z, n r. M_s, N) : A\{N/x\}
\end{align*}
\]

— $n$ is available for use in $M_s$

— Pay up front for the iteration with $D$

— Get nested iteration by passing in enough $\Diamond$s to pay for it

\[
A[n] = \Diamond^{(p_1, \ldots, p_k)}(n) \rightarrow B[n]
\]
Semantic Interpretation : Soundness
Quantitative Category with Families

1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
   Only extensional content
Quantitative Category with Families

1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
   Only extensional content
3. $U: \mathcal{L} \rightarrow \mathcal{C}$ forgetting the computational content;
Quantitative Category with Families

1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
   Only extensional content
3. $U: \mathcal{L} \to \mathcal{C}$ forgetting the computational content;
4. Addition and scaling structure on $\mathcal{L}$, fibred over $U$;
   Can add $\Gamma_1$ and $\Gamma_2$ if $U\Gamma_1 = U\Gamma_2$
Quantitative Category with Families

1. Category $L$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $C$ for interpretation contexts and simultaneous substitutions
   Only extensional content
3. $U : L \to C$ forgetting the computational content;
4. Addition and scaling structure on $L$, fibred over $U$;
   Can add $\Gamma_1$ and $\Gamma_2$ if $U\Gamma_1 = U\Gamma_2$
5. Semantic types formed with respect to $C$:
   \[ S \in Ty(\Delta), \quad \Delta \in \text{Ob} C; \]
1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
   Only extensional content
3. $U : \mathcal{L} \to \mathcal{C}$ forgetting the computational content;
4. Addition and scaling structure on $\mathcal{L}$, fibred over $U$;
   Can add $\Gamma_1$ and $\Gamma_2$ if $U\Gamma_1 = U\Gamma_2$
5. Semantic types formed with respect to $\mathcal{C}$:
   $$S \in \text{Ty}(\Delta), \quad \Delta \in \text{ObC};$$
6. Semantic terms, resourced and unresourced:
   $$M \in \text{Tm}(\Delta, S), \quad \Delta \in \text{ObC}, S \in \text{Ty}(\Delta)$$
   $$M \in \text{RTm}(\Gamma, S), \quad \Gamma \in \text{ObL}, S \in \text{Ty}(U\Gamma);$$
Quantitative Category with Families

1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content

2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
   Only extensional content

3. $U: \mathcal{L} \rightarrow \mathcal{C}$ forgetting the computational content;

4. Addition and scaling structure on $\mathcal{L}$, fibred over $U$;
   Can add $\Gamma_1$ and $\Gamma_2$ if $U\Gamma_1 = U\Gamma_2$

5. Semantic types formed with respect to $\mathcal{C}$:

   \[ S \in \text{Ty}(\Delta), \quad \Delta \in \text{ObC}; \]

6. Semantic terms, resourced and unresourced:

   \[
   M \in \text{Tm}(\Delta, S), \quad \Delta \in \text{ObC}, S \in \text{Ty}(\Delta) \\
   M \in \text{RTm}(\Gamma, S), \quad \Gamma \in \text{ObL}, S \in \text{Ty}(U\Gamma);
   \]

7. Semantic zero-ing: $U: \text{RTm}(\Gamma, S) \rightarrow \text{Tm}(U\Gamma, S)$;
Quantitative Category with Families

1. Category $\mathcal{L}$ for interpreting contexts and simultaneous substitutions
   Computational and extensional content
2. Category $\mathcal{C}$ for interpretation contexts and simultaneous substitutions
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   \[ M \in \text{RTm}(\Gamma, S), \quad \Gamma \in \text{Ob}\mathcal{L}, S \in \text{Ty}(U\Gamma); \]
7. Semantic zero-ing: $U : \text{RTm}(\Gamma, S) \rightarrow \text{Tm}(U\Gamma, S)$;
8. Resourced counterparts of substitution and comprehension, preserved by $U$. 
1. Let $\mathcal{A}$ be an $\mathcal{R}$-LCA

2. Let $C = \text{Set}$, category of sets and functions

3. Let $\mathcal{L}$ be:
   - Objects: $(|\Gamma|, \models_{\Gamma})$, where $|\Gamma|$ is a set, and $\models_{\Gamma} \subseteq \mathcal{A} \times |\Gamma|$;
   - Morphisms: $f : |\Gamma_1| \to |\Gamma_2|$, for which
     there exists an $a_f \in \mathcal{A}$ such that for all $x, a_x, a_x \models_{\Gamma_1} x$ implies $a_f \cdot a_x \models f(x)$.

4. $U : \mathcal{L} \to C$ forgets the computational information

5. Types $S \in \text{Ty}(\Delta)$ include computational information, but:
   - only depend on non-computational part

6. Terms $M \in \text{RTm}(\Gamma, S)$ are tracked by realisers from $\mathcal{A}$

7. Terms $M \in \text{Tm}(\Delta, S)$ are set theoretic functions

Read constructively, yields an “efficient” compilation method for QTT, which respects and uses the usage information.
“Length” models \((\text{Hofmann, 2003})\)

- \(\mathcal{A} = \mathbb{N}\)
- Application:
  \[ a \cdot b = a + b \]
- Combinators:
  \[
  \begin{align*}
    B &= 0 \\
    C &= 0 \\
    I &= 0 \\
  \end{align*}
  \]
- “Precious” booleans:
  \[
  \begin{align*}
    \diamond &= 1 \\
    T &= 1 \\
    F &= 1 \\
  \end{align*}
  \]
- Have to restrict interpretation of terms to always be 0
Realisability for ICC  
(Dal Lago & Hofmann, 2011)

- $\mathbb{N}_{-\infty}$ a category with objects $\mathbb{N} \cup \{-\infty\}$ and $m \rightarrow n$ if $m \leq n$, with $-\infty \leq n$
- Strict symmetric monoidal category with $(+, 0)$
- A resource monoid $M$ is a $\mathbb{N}_{-\infty}$ enriched strict symmetric monoidal category.
- Assume a model of computation with a cost model:

$$e, \eta \Downarrow_k v$$

Expressions $\mathcal{E}$ and values $\mathcal{V}$.

- Realisers are pairs $\alpha, v \in M \times \mathcal{V}$
- Functions $f: \Gamma_1 \rightarrow \Gamma_2$:
  - $f: |\Gamma_1| \rightarrow |\Gamma_2|$
  - $e \in \mathcal{E}, \gamma \in M$
  - for all $\alpha, v, a$. $\alpha, v \models_{\Gamma_1} a$ implies exists $\beta, k, v'$ s.t. $e, [v] \Downarrow_k v'$ and $\beta, v' \models_{\Gamma_2} f(a)$ and $k \leq M(\alpha + \gamma, \beta)$
Resource Monoids

For LFPL: $M \ni (n, f)$, where

- $n \in \mathbb{N}$ is the amount of iterability (number of $\Diamond$'s)
- $f$ is a polynomial with $\mathbb{N}$ coefficients
- Cost differencing:

$$M((n, f), (m, g)) = \begin{cases} g(m) - f(m) & n \leq m \text{ and } (g - f) \text{ is non-negative and non-decreasing} \\ -\infty & \text{otherwise} \end{cases}$$

- $(n, 0), * \models \Diamond *$ if $n \geq 1$. 
Resource Monoids

- For LFPL: $M \ni (n, f)$, where
  - $n \in \mathbb{N}$ is the amount of iterability (number of $\Diamond$s)
  - $f$ is a polynomial with $\mathbb{N}$ coefficients
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$$M((n, f), (m, g)) = \begin{cases} g(m) - f(m) & \text{if } n \leq m \text{ and } (g - f) \text{ is non-negative and non-decreasing} \\ -\infty & \text{otherwise} \end{cases}$$

- $(n, 0), * \models \Diamond *$ if $n \geq 1$.

- For Amortised Analysis: restrict $f$ to be degree $\leq 1$

- Cost is accounted for directly by the $\Diamond$s
Internalising Resource Reasoning?
Internalising Realisability

Intuition:

\[
\begin{align*}
0\Gamma & \vdash A \\
\therefore & \\
0\Gamma & \vdash R(A)
\end{align*}
\]

is the type of “efficiently” realisable programs of type \( A \).

For LFPL: \( R(\text{Nat} \rightarrow \text{Nat}^\circ) = \text{Poly time functions} \)
Internalising Realisability

Intuition:

\[
\begin{align*}
0\Gamma \vdash A \\
\hline
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\end{align*}
\]

is the type of “efficiently” realisable programs of type \(A\).

For LFPL: \(R(\text{Nat} \to \text{Nat}^\circ) = \text{Poly time functions}\)

**Introduction and Elimination**

\[
\begin{align*}
0\Gamma \vdash a \overset{1}{:} A \\
\hline
0\Gamma \vdash r(a) \overset{\sigma}{:} R(A)
\end{align*}
\quad
\begin{align*}
\Gamma \vdash a \overset{\sigma}{:} R(A) \\
\hline
\Gamma \vdash r^{-1}(a) \overset{\sigma}{:} A
\end{align*}
\]
Internalising Realisability

Intuition:

\[
\frac{0\Gamma \vdash A}{0\Gamma \vdash R(A)}
\]

is the type of “efficiently” realisable programs of type \( A \).

For LFPL: \( R(\text{Nat} \to \text{Nat}^\circ) = \text{Poly time functions} \)

Introduction and Elimination

\[
\begin{align*}
0\Gamma \vdash a : A & \quad & \Gamma \vdash a^\sigma : R(A) \\
0\Gamma \vdash r(a)^\sigma : R(A) & \quad & \Gamma \vdash r^{-1}(a)^\sigma : A
\end{align*}
\]

E.g.

\[
\frac{0\Gamma \vdash M : (x : A) \to B}{0\Gamma \vdash r(M) : R((x : A) \to B)}
\]

means “the function \( M \) is realisable in the underlying computational model”. 
Internalising Realisability

Interpretation:

\[ \left\{ a \in \gamma \mid \exists x \in A. x = A(\gamma) a \right\} \]
Internalising Realisability

Interpretation:

\[ \sem{[R(A)]} \gamma = \{ a \in \sem{A} \gamma \mid \exists x \in A. x \models_{A(\gamma)} a \} \]

**Internalise** the realisability:

\[
R(A) = \exists a^0 : A. \exists p^0 : \text{Prog}. p \models_{A} a
\]

with definition of \( \models_{A} \) above, capture polytime functions as a type.
Internalising Realisability

Interpretation:

\[
\llbracket R(A) \rrbracket_\gamma = \{ a \in \llbracket A \rrbracket_\gamma \mid \exists x \in \mathcal{A}. x \models_{A(\gamma)} a \}\]

**Internalise** the realisability:

\[
R(A) = \exists a^0 : A. \exists p^0 : \text{Prog}. p \models_{A} a
\]

with definition of \(\models_{A}\) above, capture polytime functions as a type.

Note: \(R(A)\) must be a proposition – cannot allow intensional information to be used to make decisions in the extensional world!
Summary
Quantitative Type Theory:
Fixed and extended formulation of McBride’s “Plenty o’ Nuttin’ ” system

Quantitative Type Theory for ICC
Dependently typed version of LFPL and Amortised Analysis

Categorical and Realisability models

Goal: to combine resource-ful and resource-less programming and reasoning.
Quantitative Type Theory:
   Fixed and extended formulation of McBride’s “Plenty o’ Nuttin’” system

Quantitative Type Theory for ICC
   Dependently typed version of LFPL and Amortised Analysis

Categorical and Realisability models

Goal: to combine resource-ful and resource-less programming and reasoning.

Related Work

- Sized types
  Used for controlling well foundedness
  For complexity analysis require “tick” monads

- Gaboardi and Dal Algo: Linear Dependent Types for ICC
  Dependent Types only for counting time

Future:

- Soft linear logic, LAL, EAL
- Polytime mathematics?