Syntax and Semantics of Quantitative Type Theory

Robert Atkey Strathclyde University, Glasgow robert.atkey@strath.ac.uk

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In Martin-Löf Type Theory

 $x_1:S_1,\ldots,x_n:S_n\vdash M:T$

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variables x_1, \ldots, x_n are mixed usage

x is used *computationally*

n is used *logically*

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In Linear Logic

$$x_1:X_1,\ldots,x_n:X_n\vdash M:Y$$

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presence of a variable records usage each x_i must be used by M exactly once

$n : \mathsf{Nat}, x : \mathsf{Fin}(n) \vdash x : \mathsf{Fin}(n)$

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Is *n* even used at all?

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will come back to this...

$$n : \text{Nat} \mid x : \text{Fin}(n) \vdash x : \text{Fin}(n)$$

- ▶ Separate *intuitionistic* / *unrestricted* uses and *linear* uses
- ▶ Types can depend on intuitionistic data, but not linear data

will come back to this...

```
(Barber, 1996)
(Cervesato and Pfenning, 2002)
(Krishnaswami, Pradic, and Benton, 2015)
(Vákár, 2015)
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Separation interferes with dependency:

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$$n : \text{Nat}, x : \text{Fin}(n) \mid \hat{x} : \hat{\text{Fin}}(n, x) \vdash (x, \hat{x}, \text{refl}(x)) : (y : \text{Fin}(n)) \times \hat{\text{Fin}}(n, y) \otimes (x \equiv y)$$

Quantitative Coeffect calculi:

$$x_1 \stackrel{\rho_1}{:} S_1, \ldots, x_n \stackrel{\rho_n}{:} S_n \vdash M : T$$

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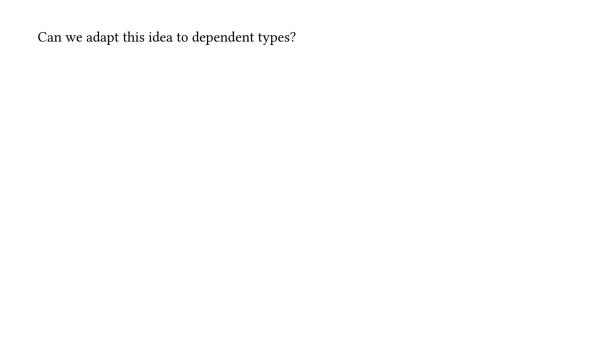
- \triangleright The ρ_i record usage from some semiring R
 - $1 \in R a$ use
 - $0 \in R \text{not used}$
 - . $\rho_1 + \rho_2$ adding up uses (e.g., in an application)
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(Petricek, Orchard, and Mycroft, 2014) (Brunel, Gaboardi, Mazza, and Zdancewic, 2014) (Ghica and Smith, 2014)



Can we adapt this idea to dependent types?

McBride's idea:

 $\, \triangleright \,$ allow 0-usage data to appear in types.

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$$x_1 \stackrel{\rho_1}{:} S_1, \ldots, x_n \stackrel{\rho_n}{:} S_n \vdash M \stackrel{\sigma}{:} T$$

where $\sigma \in \{0, 1\}$.

 $\triangleright \sigma = 1$ — the "real" computational world

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(allowing arbitrary ρ yields a system where substitution is inadmissible)

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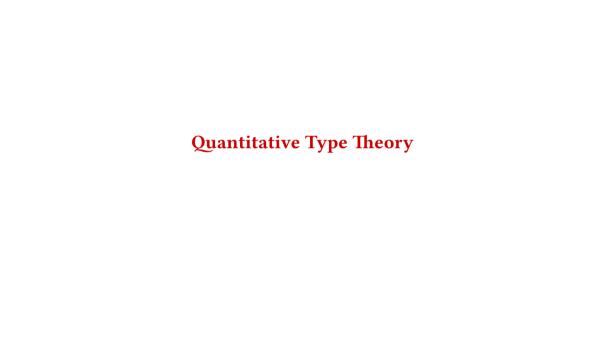
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Zero-ing is an admissible rule: $\frac{\Gamma \vdash M \stackrel{1}{:} T}{0\Gamma \vdash M \stackrel{0}{:} T}$ allowing promotion to the type world.



Quantitative Type Theory

Contexts

$$\frac{\Gamma \vdash 0\Gamma \vdash S}{\Gamma, x : S \vdash} \text{ Ext}$$

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$$\frac{}{\Leftrightarrow \vdash} \ _{\mathsf{EMP}} \qquad \qquad \frac{\Gamma \vdash \quad 0\Gamma \vdash S}{\Gamma, x \stackrel{\rho}{:} S \vdash} \ _{\mathsf{Ext}}$$

Types

$$0\Gamma \vdash S$$

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Types

$$0\Gamma \vdash S$$

Terms

$$\frac{0\Gamma, x \overset{\sigma}{:} S, 0\Gamma' \vdash}{0\Gamma, x \overset{\sigma}{:} S, 0\Gamma' \vdash x \overset{\sigma}{:} S} \text{Var} \qquad \frac{\Gamma \vdash M \overset{\sigma}{:} S \qquad 0\Gamma \vdash S \equiv T}{\Gamma \vdash M \overset{\sigma}{:} T} \text{Conv}$$

 Π -type formation

$$\frac{0\Gamma \vdash S \qquad 0\Gamma, x \overset{0}{:} S \vdash T}{0\Gamma \vdash (x \overset{\pi}{:} S) \to T}$$

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 Π -type introduction and elimination

$$\frac{\Gamma, x \stackrel{\sigma\pi}{:} S \vdash M \stackrel{\sigma}{:} T}{\Gamma \vdash \lambda x \stackrel{\pi}{:} S.M^T \stackrel{\sigma}{:} (x \stackrel{\pi}{:} S) \to T}$$

$$\frac{\Gamma_1 \vdash M \stackrel{\sigma}{:} (x \stackrel{\pi}{:} S) \to T \qquad \Gamma_2 \vdash N \stackrel{\sigma'}{:} S \qquad 0 \Gamma_1 = 0 \Gamma_2 \qquad \sigma' = 0 \Leftrightarrow (\pi = 0 \lor \sigma = 0)}{\Gamma_1 \vdash M \stackrel{\sigma}{:} (x \stackrel{\pi}{:} S) \to T \qquad \Gamma_2 \vdash N \stackrel{\sigma'}{:} S \qquad 0 \Gamma_1 = 0 \Gamma_2 \qquad \sigma' = 0 \Leftrightarrow (\pi = 0 \lor \sigma = 0)}$$

$$\Gamma_1 + \pi \Gamma_2 \vdash \operatorname{App}_{(x^{\pi}S)T}(M, N) \stackrel{\sigma}{:} T[N/x]$$

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There are three cases:

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There are three cases:

- 1. $\sigma = 1, \pi \neq 0, \sigma' = 1$ Function uses its argument "computationally";
- 2. $\sigma = 1, \pi = 0, \sigma' = 0$ Function yields computational data, but doesn't use its argument;

$$\frac{\Gamma_1 \vdash M \overset{\sigma}{:} (x \overset{\pi}{:} S) \to T \qquad \Gamma_2 \vdash N \overset{\sigma'}{:} S \qquad 0 \\ \Gamma_1 + \pi \Gamma_2 \vdash \mathrm{App}_{(x \overset{\pi}{:} S)T}(M, N) \overset{\sigma}{:} T[N/x]}$$

There are three cases:

- 1. $\sigma = 1, \pi \neq 0, \sigma' = 1$ Function uses its argument "computationally";
- **2.** $\sigma = 1, \pi = 0, \sigma' = 0$ Function yields computational data, but doesn't use its argument;
- 3. $\sigma = 0, \sigma' = 0$ We are not using the function for its computational content, so π does not matter.

In the paper... additional type formers

- **1.** Quantitative Σ -types;
- 2. Boolean type;
- 3. Universe of small sets (only exists in the $\sigma = 0$ fragment).

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Weakening

$$\frac{\Gamma, \Gamma' \vdash \mathcal{J} \qquad 0\Gamma \vdash U}{\Gamma, x \stackrel{0}{:} U, \Gamma' \vdash \mathcal{J}} \text{ Weaken}$$

Zero-ing

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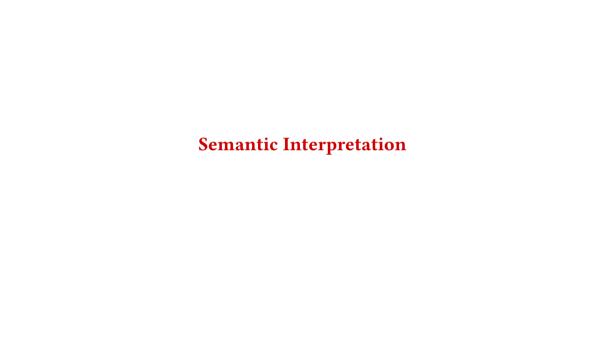
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Substitution

$$\frac{\Gamma_1, x \stackrel{\rho}{:} S, \Gamma' \vdash M \stackrel{\sigma}{:} T \qquad \Gamma_2 \vdash N \stackrel{\sigma'}{:} S \qquad 0\Gamma_1 = 0\Gamma_2 \qquad \sigma' = 0 \Leftrightarrow \rho = 0}{(\Gamma_1 + \rho \Gamma_2), \Gamma'[N/x] \vdash M[N/x] \stackrel{\sigma}{:} T[N/x]}$$



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5. Semantic terms, resourced and unresourced:

$$M \in \operatorname{Tm}(\Delta, S), \quad \Delta \in \operatorname{Ob}C, S \in \operatorname{Ty}(\Delta)$$

 $M \in \operatorname{RTm}(\Gamma, S), \quad \Gamma \in \operatorname{Ob}\mathcal{L}, S \in \operatorname{Ty}(U\Gamma);$

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6. Semantic zero-ing: $U : \operatorname{RTm}(\Gamma, S) \to \operatorname{Tm}(U\Gamma, S)$;

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- **6.** Semantic zero-ing: $U: RTm(\Gamma, S) \to Tm(U\Gamma, S)$;
- 7. Resourced counterparts of substitution and comprehension, preserved by U.

Realisability Models

R-Linear Combinatory Algebras (R-LCAs)

(Abramsky, Haghverdi, and Scott, 2002)

A carrier $\mathcal A$ with an application operator (\cdot) and unary operators $!_{\rho}$, $\rho \in R$, and:

$$B \cdot x \cdot y \cdot z = x \cdot (y \cdot z)$$

$$C \cdot x \cdot y \cdot z = x \cdot z \cdot y$$

$$I \cdot x = x$$

$$K \cdot x \cdot !_{0}y = x$$

$$W_{\pi\rho} \cdot x \cdot !_{\pi+\rho}y = x \cdot !_{\pi}y \cdot !_{\rho}y$$

$$D \cdot !_{1}x = x$$

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$$F_{\rho} \cdot !_{\rho}x \cdot !_{\rho}y = !_{\rho}(x \cdot y)$$

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For example: any LCA gives a 0, 1, ω -LCA; any BCI algebra gives a \mathbb{N} -LCA; any SK algebra gives a $\{0,1\}$ -LCA.

Realisability Models

- **1.** Let C = Set, category of sets and functions
- **2.** Let \mathcal{L} be Assemblies over \mathcal{A} sets with computational information
- **3.** $U: \mathcal{L} \to C$ forgets the computational information
- **4.** Types $S \in \text{Ty}(\Delta)$ include computational information, but: only depend on non-computational part
- **5.** Terms $M \in \text{RTm}(\Gamma, S)$ are tracked by realisers from \mathcal{A}
- **6.** Terms $M \in \text{Tm}(\Delta, S)$ are set theoretic functions

Read constructively, yields an "efficient" compilation method for QTT, which respects and uses the usage information.

Main contribution

An interpretation of Type Theory that allows for "sub-computational" models

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An interpretation of Type Theory that allows for "sub-computational" models

- **1.** No longer need the full power of SK;
- **2.** *R*-LCAs allow for interesting refined models of computation;
- **3.** For example (future work!)
 - 3.1 Hofmann and Dal Lago's realisability models of Implicit Computational Complexity;
 - **3.2** Geometry of Interaction models, incl Abramsky's *A structural approach to reversible computation, 2005*;
 - **3.3** Imperative models, after Ahmed, Fluet, Morrisett's L³;
 - **3.4** Staged models



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Future work

- ▶ More *R*-LCAs, for more applications (ICC, reversible computation, ...)
- Combination with split context systems
- Implementation (already started in Idris)
- ▶ Internalisation of extensional properties yielded by quantitative information

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Thank You for Listening!