# Theorems for Free

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#### **Deduce** the following "Free Theorem":

for all types X, Y, for all functions  $f: X \to Y$ , for all lists l: List X,  $M \pmod{f l} = \max f(M l)$ .

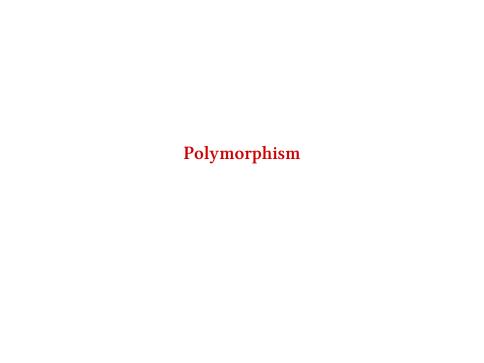
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Without looking at the implementation of M.



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Parametric Polymorphism

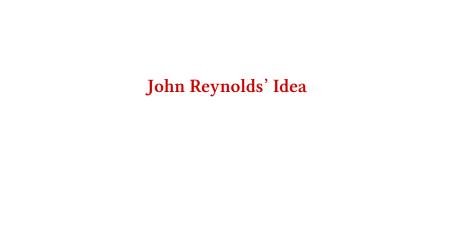
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#### Parametric Polymorphism

$$\forall \alpha. \text{ List } \alpha \rightarrow \text{List } \alpha$$
  $\}$  single implementation

single implementation works for all types



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#### From Types to Relations

If A is a type with free variables  $\alpha_1, ..., \alpha_n$ and we have types  $X_1, ..., X_n$ , and  $Y_1, ..., Y_n$ and relations  $R_1: X_1 \leftrightarrow Y_1, ..., R_n: X_n \leftrightarrow Y_n$ 

then  $|A|: (A[X_1/\alpha_1,...,X_n/\alpha_n]) \leftarrow (A[Y_1/\alpha_1,...,Y_n/\alpha_n])$ 

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# Definition

$$\begin{array}{lll} \left[\alpha_{i}\right] & = & R_{i} \\ \left[\forall\alpha.A\right] & = & \left\{(x,x') \mid \forall X,Y,R:X \leftrightarrow Y.\ x \lfloor A \rfloor x'\right\} \\ \left[A \rightarrow B\right] & = & \left\{(f,g) \mid \forall (x,x') \in \lfloor A \rfloor.\ (fx,g\ x') \in \lfloor B \rfloor\right\} \\ \left[\text{List } A\right] & = & \left\{(l,l') \mid |l| = |l'| \text{ and for all } i,\ l_{i}\lfloor A \rfloor l'_{i}\right\} \end{array}$$

# Reynolds' Abstraction Theorem

If M has type A, then  $M \lfloor A \rfloor M$ .

#### Reynolds' Abstraction Theorem

If *M* has type *A*, then M|A|M.

#### Caveats

Depends heavily on the language!

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general recursion : (some) relations must be admissible effects : (some) relations must be  $\top \top$ -closed

seq : (some) relations must be bottom-reflecting

typecase : things get weird

#### $M: \forall \alpha. \ \mathsf{List} \ \alpha \to \mathsf{List} \ \alpha$

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#### Which means

For all types X, Y and relations  $R: X \leftrightarrow Y$ , for all  $l_X$  and  $l_Y$  such that  $l_X \lfloor \text{List } \alpha \rfloor l_Y$ ,  $(M l_X) \rfloor \text{List } \alpha \rfloor (M l_Y)$ 

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## Instantiating

Set  $R = \{(x, y) \mid fx = y\}$ then  $l_X \lfloor \text{List } \alpha \rfloor l_Y \Leftrightarrow l_Y = \text{map } f l_X$ so for all  $l_X \mid M \mid \text{map } f \mid l_X \mid M \mid l_X \mid l$ 

# Representing Datatypes

$$\forall \alpha. \ \alpha \to (A \to \alpha \to \alpha) \to \alpha$$
 $\cong$ 
 $\mathsf{List}(A)$ 

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 $\lambda$ nil.  $\lambda$ cons. cons  $a_1$  (cons  $a_2$  nil)  $\approx [a_1, a_2]$ 

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#### Representing Syntax

$$\forall \alpha. ((\alpha \to \alpha) \to \alpha) \to (\alpha \to \alpha \to \alpha) \to \alpha$$
  
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Terms of the untyped  $\lambda$ -calculus with no free type variables

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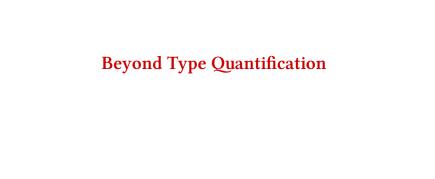
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Terms of the untyped  $\lambda$ -calculus with no free type variables

 $\lambda$ lam.  $\lambda$ app. lam ( $\lambda x$ . lam ( $\lambda y$ . app x y))  $\approx \lambda x y . x y$ 



$$area: \forall B: \mathsf{GL}_2, t: \mathsf{T}_2.$$

$$\mathsf{vec}\langle B, t \rangle \to \mathsf{vec}\langle B, t \rangle \to \mathsf{vec}\langle B, t \rangle \to \mathsf{real}\langle |\det B| \rangle$$

$$\begin{array}{c} \mathit{area} : \forall \mathit{B} : \mathsf{GL}_2, \mathit{t} : \mathsf{T}_2. \\ & \mathsf{vec} \langle \mathit{B}, \mathit{t} \rangle \rightarrow \mathsf{vec} \langle \mathit{B}, \mathit{t} \rangle \rightarrow \mathsf{vec} \langle \mathit{B}, \mathit{t} \rangle \rightarrow \mathsf{real} \langle | \mathsf{det} \, \mathit{B} | \rangle \end{array}$$

#### **Deduce** translation invariance:

$$\forall \vec{t}, \vec{v_1}, \vec{v_2}, \vec{v_3}$$
. area  $(v_1+t)$   $(v_2+t)$   $(v_3+t)=$  area  $v_1$   $v_2$   $v_3$ 

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#### *Deduce* orthogonal transformation invariance:

$$\forall O, \vec{v_1}, \vec{v_2}, \vec{v_3}. \ area \ (Ov_1) \ (Ov_2) \ (Ov_3) = area \ v_1 \ v_2 \ v_3$$

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**Deduce** scaling variance:

$$\forall s, \vec{v_1}, \vec{v_2}, \vec{v_3}. \ area \ (s \cdot v_1) \ (s \cdot v_2) \ (s \cdot v_3) = s^2 \cdot area \ v_1 \ v_2 \ v_3$$

# Types to Relations

```
    \begin{bmatrix} \forall B : \mathsf{GL}_2.A \end{bmatrix} = \{(x, x') \mid \forall B : \mathsf{GL}_2. \ x \lfloor A \rfloor x' \} 

    \begin{bmatrix} \forall t : \mathsf{T}_2.A \end{bmatrix} = \{(x, x') \mid \forall t : \mathsf{T}_2. \ x \lfloor A \rfloor x' \} 

    |\forall s : \mathsf{GL}_1.A | = \{(x, x') \mid \forall s : \mathsf{GL}_1. \ x | A | x' \}
```

*area* :  $\forall B: \mathsf{GL}_2, t: \mathsf{T}_2$ .

 $ext{vec}\langle B,t 
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Symmetry as a Guide:

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select one to be the "origin":

$$(p_2-p_1): \mathtt{vec}\langle \mathit{B}, \mathit{0} \rangle \text{ and } (p_3-p_1): \mathtt{vec}\langle \mathit{B}, \mathit{0} \rangle$$

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remove rotational symmetry and get area of parallelogram:

$$(p_2 - p_1) \times (p_2 - p_1)$$
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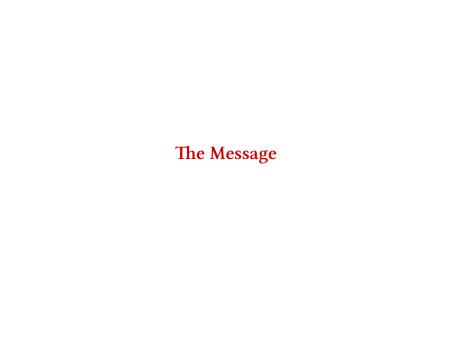
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#### Polymorphism means Uniformity

If a type has a  $\forall$ , then implementations are uniform under change:

 $\forall \alpha$ . : uniform under change of data representation

 $\forall B: \mathsf{GL}_2.$  : uniform under change of basis

 $\forall t : \mathsf{T}_2$ . : uniform under change of origin

 $\forall s : \mathsf{GL}_1$ . : uniform under change of scale

Uniformity allows one to deduce "free theorems"

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What other sorts of uniformity are useful?